Rapport de stage de recherche
Fundamental gap δ property

Thomas Giarrizzi
Encadrante : Guofang Wei
Lieu : University of California, Santa Barbara

Si l'on m'avait demandé, peu avant mon départ, d'imaginer le déroulement idéal d'une première expérience de recherche en terrain inconnu, j'aurais sans doute décrit une situation très proche de ce qui s'est effectivement produit.

Déroulement du stage

Après avoir longuement réfléchi, en concertation avec ma tutrice, Madame Eleonora Di Nezza, sur le lieu idéal pour effectuer mon stage de deuxième année, j'ai vu mon "rêve" prendre forme : partir aux États-Unis. Destination : l'Université de Californie à Santa Barbara (UCSB), où j'ai eu l'opportunité de travailler sous la direction de Madame Guofang Wei, professeure éminente en géométrie différentielle et en analyse géométrique.

Mon stage, prévu sur trois mois – d'avril à juin – a commencé par un véritable parcours du combattant administratif pour obtenir le fameux visa J-1. Mais le 31 mars 2025, c'est le grand jour : j'arrive à Los Angeles, le cœur battant, avant de rejoindre Santa Barbara dès le lendemain. Sur place, l'accueil est à la hauteur de mes espérances. Madame Guofang Wei me fait visiter le département de mathématiques : les salles communes, les espaces de travail et le bureau que je partagerai avec une collègue. Rapidement, je choisis de passer le plus clair de mon temps à la grande bibliothèque universitaire, ouverte presque 24h/24, afin de m'immerger totalement dans la vie étudiante et de ne pas déranger ma camarade de bureau.

Chaque mardi, à 14 h, un rendez-vous hebdomadaire me permettait de faire le point sur mes projets en cours. Je posais des questions larges, parfois volontairement vagues, pour pouvoir creuser moi-même les pistes intéressantes ou en écarter d'autres.
Ce qui m'a particulièrement marqué, ce fut la liberté quasi absolue dont je disposais, tant sur le fond — à savoir le choix du sujet à traiter et des problèmes à approfondir — que sur la forme : je travaillais quand je le souhaitais, où bon me semblait et avec qui je jugeais pertinent.
S'il ne m'est pas particulièrement inconfortable de travailler avec beaucoup de directives, cela peut néanmoins produire chez moi un effet secondaire non négligeable : celui de se contenter du strict nécessaire, sans réelle volonté de se dépasser.

En l'absence de règles, la mécanique se met en marche d'elle-même, et le plaisir de l'indépendance se fait immédiatement sentir, ce qui a été le cas ici, comme mentionné plus haut. Je travaillais finalement entre 60 et 70 heures par semaine — ce que l'on pourrait qualifier de mon rythme de croisière — en continuant de ne pas distinguer les jeudis des dimanches.

Parallèlement, j'ai eu l'opportunité de suivre le cours *Topics in differential geometry*, dispensé par le professeur Matthias Wink, ainsi que de participer régulièrement au séminaire hebdomadaire consacré à la géométrie. Et pour couronner le tout : j'ai également eu la chance de prendre part à trois écoles d'été, dont les thématiques étaient toutes, à des degrés divers, en lien avec mes travaux de recherche :

- 17/06/25-21-06/25 : *Optimal transport, heat flow and synthetic Ricci bounds, in honor of Luigi Ambrosio*, Northwestern University, Chicago, United States.
- 13/07/25-26/07/25 : *Geometric Measure Theory*, Hangzhou, China.
- 28/07/25-01/08/25 : *Interations between Convex Geometry and Spectral Analysis*, Montréal, Canada.

Remerciements

Dans un premier temps, qu'il me soit permis d'adresser mes plus sincères remerciements à Madame Guofang Wei, qui a bien voulu, avec une grande bienveillance, accepter de me superviser au cours de ces derniers mois. Elle m'a accordé une autonomie précieuse tout au long de ce stage, ce qui a représenté un véritable privilège. C'est également grâce à son appui que j'ai pu bénéficier d'un financement partiel me permettant de participer à la première École d'été, organisée à Chicago. Je tiens également à remercier l'ENS, et plus particulièrement le département de mathématiques, pour le soutien financier partiel accordé dans le cadre de mon stage — dont les frais se sont révélés relativement conséquents — ainsi que pour l'efficacité et la disponibilité constantes de son administration. Enfin, je souhaiterais remercier Madame Eleonora Di Nezza, grâce à qui j'ai pu entrer en contact avec Madame Guofang Wei, et qui m'a également offert l'opportunité de participer à deux autres écoles d'été à l'étranger.

Contents

0 Backgrounds and Notation 3

1 Introduction 4

2 Fundamental gap of the Euclidean space and the round sphere 6
- 2.1 Modulus of continuity/expansion . . . 6
- 2.2 Spectrum comparison . . . 7

3 Stability of Γ_δ under Gromov-Hausdorff convergence 9
- 3.1 Gromov-Hausdorff convergence . . . 9
- 3.2 Caratheodory and GCE Properties . . . 10
- 3.3 Convergence of convex subsets . . . 12
- 3.4 Spectrum convergence . . . 14

4 Lower-semicontinuous conditions **15**
4.1 Uniformly Equivalent Metrics 15
4.2 CTL property 16

0 Backgrounds and Notation

Throughout this paper, a pair (M^n, g) will denote a *Riemannian manifold*; that is, an n-dimensional smooth M^n manifold with a Riemannian metric g, i.e, a smooth symmetric positive definite $(0,2)$-tensor field on M^n. We will assume that M^n is connected. The metric g induces an inner product

$$g_p : (X, Y) \in T_pM \times T_pM \longmapsto g_p(X, Y) \in \mathbb{R}.$$

The *Levi-Civita connection* ∇ on (M^n, g) is the unique torsion-free affine connection that is compatible with the metric, meaning that $\nabla g = 0$. For vector fields X, Y, Z on M^n, it satisfies

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The *Riemannian distance* $d_g(p, q)$ between two points $p, q \in M^n$ is defined as the infimum of the lengths of piecewise smooth curves connecting p and q, where the length of a curve $\gamma : [a, b] \to M^n$ is

$$l(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt.$$

A *geodesic* is a curve γ that locally minimizes the distance, satisfying the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$. A geodesic is *minimizing* if it achieves the infimum distance between its endpoints.

A subset $\Omega \subset M^n$ is *convex* if for any two points $x, y \in \Omega$, any minimizing geodesic between x and y lies in Ω.

The *volume form* associated to g is denoted by $d\mathrm{Vol}_g$ or dV_g, which in local coordinates $(x^1, \dots, x^n)$ is

$$dVol_g(x) = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n, \text{ where } g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j).$$

The *gradient* of a smooth function $f : M^n \to \mathbb{R}$ is the vector field $\nabla_g f$ defined by $g(\nabla_g f, X) = df(X)$ for any vector field X.

The *Laplacian* (or Laplace-Beltrami operator) Δ_g acting on smooth functions f is defined as $\Delta_g f = \mathrm{div}_g(\nabla_g f)$, where the divergence of a vector field X is $\mathrm{div}_g X = \mathrm{trace}_g(\nabla X)$.

The *Ricci curvature* Ric_g is the $(0,2)$-tensor defined by

$$\mathrm{Ric}_g(X, Y) = \mathrm{trace}_g(Z \mapsto R(Z, X)Y),$$

where R is the Riemann curvature tensor given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The *sectional curvature* (denoted sec_g) in the plane spanned by orthonormal vectors X, Y is $g(R(X, Y)Y, X)$.

1 Introduction

Given a bounded smooth domain $\Omega \subset M^n$ of a Riemannian manifold, the eigenvalue equation of the Laplacian on Ω is $\Delta u = -\lambda u$. We may impose boundary conditions, which will be the subject of further discussion :

Dirichlet conditions

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Neumann conditions

$$\begin{cases} \Delta v = -\mu v & \text{in } \Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega \end{cases}$$

Thanks to classical results on compact self-adjoint operators, the eigenvalues consist of an infinite sequence going off to infinity :

$$\textbf{Dirichlet } (u|_{\partial\Omega} = 0) : \quad 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty$$

$$\textbf{Neumann } (\partial_n v|_{\partial\Omega} = 0) \text{ / closed manifolds} : \quad 0 = \mu_0 < \mu_1 \leq \mu_2 \cdots \to \infty$$

and we have the min-max principle to characterize each eigenvalue

$$\lambda_k(\Omega) = \min_{\substack{V \subset H_0^1(\Omega) \\ \dim V = k}} \max_{\substack{f \in V \\ f \neq 0}} R_g[f] \qquad \mu_k(\Omega) = \min_{\substack{V \subset H^1(\Omega) \\ \dim V = k+1}} \max_{\substack{f \in V \setminus \{0\} \\ \int_\Omega f = 0}} R_g[f]$$

with the Rayleigh quotient $R_g[f] = \dfrac{\int_\Omega \|\nabla^g f\|_g^2 dV_g}{\int_\Omega f^2 dV_g}$. Due to the characterization above, we have the decreasing monotonicity of Dirichlet eigenvalues under domain inclusion, i.e for two domains Ω_1 and Ω_2, if $\Omega_1 \subset \Omega_2$, then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$. One of the central problems is to estimate the so-called **fundamental gap**, i.e the quantity $\lambda_2(\Omega) - \lambda_1(\Omega)$. Since except for some very specific domains (such as Euclidean balls, rectangles and triangles, spherical triangles and lunes, etc...), it is extremely difficult to compute the eigenvalues explicitly, we need to provide some optimal geometric lower and upper bounds of the gap. This subject has a long story, and most available results are established in the Euclidean setting. For example, in Euclidean space, there was a conjecture stated by M. van den Berg in 1983 after some observations on some particular cases, saying that for each convex domain $\Omega \subset \mathbb{R}^n$, one has

$$\lambda_2(\Omega) - \lambda_1(\Omega) \geq \frac{3\pi^2}{D(\Omega)^2},$$

where $D(\Omega)$ represents the diameter of the domain Ω. The bound $\frac{\pi^2}{4D(\Omega)^2}$ was firstly found by Singer-Wong-Yau-Yau in [6], followed by the bound $\frac{\pi^2}{D(\Omega)^2}$ found by Yu-Zhong [7], Ling [8]. Finally, in 2011, the Fundamental Gap Conjecture was finally resolved by Andrews-Clutterbuck [9], and the bound $\frac{3\pi^2}{D(\Omega)^2}$ is optimal and is approached when the domain is a thin rectangular box. This is the subject of part 2.

This suggests a study of the same type of bound for other spaces and, in particular, for spaces with constant sectional curvature. For the round sphere $\mathbb{S}^n$, we can obtain the same bound as in the Euclidean space for $n \geq 2, D(\Omega) < \pi$. For the hyperbolic space $\mathbb{H}^n$ (and more generally for spaces with curvature not everywhere positive ≥ 0) the behavior is completely different; the gap can be made arbitrarily small, and thus there exist no strict positive lower bounds in those spaces. More precisely, we have the following result

Theorem 1.1 (G.Khan and X.H. Nguyen, 2024, [11]). *Let (M^n, g) be a smooth Riemannian manifold. Suppose that there is a point p and a tangent plane V at p satisfying $\kappa(V) < 0$, where κ is the sectional curvature. Then, for all $\varepsilon > 0$, there is a domain Ω which is geodesically convex and such that*

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\varepsilon}{D^2},$$

where D is the diameter of Ω.

We can thus introduce the following definition :

Definition 1.2 (Γ_δ property). *A space M satisfies the fundamental gap δ (denoted Γ_δ, with $\delta > 0$) property if for all convex domains Ω, the following estimate holds:*

$$\lambda_2(\Omega) - \lambda_1(\Omega) \geq \frac{\delta\pi^2}{D^2(\Omega)},$$

where $D(\Omega)$ is the diameter of Ω.
In other words, a space satisfies Γ_δ if the following inequality holds:

$$\inf\left\{(\lambda_2(\Omega) - \lambda_1(\Omega))D^2(\Omega) \,\middle|\, \Omega \subset M \text{ convex}\right\} \geq \delta\pi^2.$$

This property has been introduced in [4]. It may be interesting to see what the constant δ depends on. Actually, for a fixed domain, eigenvalues and eigenfunctions depend on the metric in a continuous way (in a smooth enough topology), so it may be tempting to conclude that since the round sphere satisfies Γ_3, a nearly-round sphere automatically satisfies Γ_1, simply by continuity. But as explained in [4], where has been introduced the Γ_δ property, we can deform the Euclidean space in C^∞ fashion to a metric with some negative sectional curvature, which thus does not satisfy the Γ_δ property for any $\delta > 0$. So the constant δ does not depend on the metric in a continuous way. We can also already deduce that the Γ_δ property gives automatically $\mathrm{Ric} \geq 0$. In the same paper, the authors posed two questions that motivated further investigation on my part :

Question 1. *Under what conditions is the Γ_δ condition deformation stable, and in what topology? For instance, do sufficiently C^2-small deformations of a round sphere satisfy $\Gamma_{2.99}$?*

Question 2. *Let (M_i, g_i) be a sequence of Riemannian manifolds which all satisfy Γ_δ and converge to an Alexandrov space M_∞. Does (M_∞, g_∞) also satisfy Γ_δ?*

In this paper, for question 2, we give a partial answer (or rather a weaker) result, by adding two more properties for each M_i to make possible the stability of Γ_δ, this is the subject of section 3. For question 1, it's more complicated; we can try to generalize the question : Can we find some conditions on manifolds to make the constant δ lower-semicontinuous with respect to a certain topology ?
And honestly, let us say it right away: the result is rather disappointing. The conditions we found are purely abstract, and it is not even clear whether such spaces actually exist — all the more so since the proof remains quite elementary given the nature of the conditions; this is the subject of section 4. Let us begin, in the next section, with a brief survey of the powerful results on the fundamental gap in Euclidean space and on the sphere.

2 Fundamental gap of the Euclidean space and the round sphere

The Euclidean space and the round sphere satisfy both Γ_3. And the techniques used to achieve the result are similar : using modulus of continuity, establishing the super log-concavity of the first eigenfunction and using two-point maximal principle. In this part, we provide a brief summary of the proof for the Euclidean case, from Andrews-Clutterbuck [9], outlining only the essential arguments. The conjecture is generalized and solved for the operator $\Delta + V$, with a convex function V. Let's denote ϕ_1 (resp. $\tilde{\phi}_1$) the first eigenfunction of the operator $-\Delta + V$ (resp. one dimensional operator $-\frac{d^2}{dx^2} + \tilde{V}$ on the interval $[-D(\Omega)/2, D(\Omega)/2]$).

2.1 Modulus of continuity/expansion

In the first article, the authors introduce few more relevant notions in a general context: Here we say that a function ω on $\mathbb{R}_+$ is

- a *modulus of continuity* of a function f on Ω if for all $x, y \in \Omega$,

$$|f(x) - f(y)| \leq 2\omega\left(\frac{|y-x|}{2}\right),$$

- a *modulus of expansion* of a vector field X if for all $x \neq y \in \Omega$,

$$(X(y) - X(x)) \cdot \frac{y-x}{|y-x|} \geq 2\omega\left(\frac{|y-x|}{2}\right),$$

- a *modulus of contraction* of a vector field X if $-\omega$ is a modulus of expansion of -X

- a *modulus of convexity* (resp. *modulus of concavity*) of a semi-convex function $f : \Omega \longrightarrow \mathbb{R}$ if ω is a modulus of expansion (resp. contraction) of the gradient vector field ∇f.

Actually, modulus of concavity/convexity are very useful to "compare" the convexity and the concavity of two functions and even to compare the spectrum of two operators. In a sense, we can say that the potential V is "more convex" than $\tilde{V}$ if $\tilde{V}$ is a modulus of convexity for V, i.e for any $x \neq y \in \Omega$ we have

$$\nabla V(y) - \nabla V(x)) \cdot \frac{(y-x)}{|y-x|} \geq 2\tilde{V}\left(\frac{|y-x|}{2}\right). \tag{1}$$

One of the most technical result is to show that the inequality 1 implies a (super) log-concavity estimate for the first eigenfunction ϕ_1 :

$$\nabla \ln \phi_1(y) - \nabla \ln \phi_1(x)) \cdot \frac{y-x}{|y-x|} \leq 2(\ln \tilde{\phi}_1)'_{z=\frac{|y-x|}{2}}, \tag{2}$$

for any $x \neq y \in \Omega$, and thus, in a sense, ϕ_1 is "more log-concave" than $\tilde{\phi}_1$. Due to the length request of this paper, we cannot even give a summary of the proof of 2 without forgetting something important. We can give instead the proof that inequality 2 solves the conjecture. Indeed, we will show that the spectral gap for $-\Delta + V$ is bounded below by the spectral gap of the one dimensional operator $-\frac{d^2}{dx^2} + \tilde{V}$. In other words, if $\tilde{\lambda}_i$ is the eigenvalue associated to $\tilde{\phi}_i$, then

$$\lambda_2(\Omega) - \lambda_1(\Omega) \geq \tilde{\lambda}_2 - \tilde{\lambda}_1. \tag{3}$$

And since the conjecture is already settled for $n = 1$ (thanks to Lavine), we can thus deduce the result for $n \geq 2$.

2.2 Spectrum comparison

Proof of inequality 2 $\Longrightarrow$ inequality 3. Let's start the proof by giving a very useful theorem (see [9]):

Theorem 2.1. *Let Ω be a strictly convex domain of diameter D with smooth boundary in $\mathbb{R}^n$, and X a time-dependent vector field on Ω. Suppose $v : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a smooth solution of the equation*

$$\frac{\partial v}{\partial t} = \Delta v + X \cdot \nabla v \quad \textit{in } \Omega \times \mathbb{R}_+; \tag{4}$$

$$D_\nu v = 0 \quad \textit{in } \partial\Omega \times \mathbb{R}_+.$$

Suppose that:

1. *$X(\cdot, t)$ has modulus of contraction $\omega(\cdot, t)$ for each $t \geq 0$, where $\omega : [0, D/2] \times \mathbb{R}_+ \to \mathbb{R}$ is smooth;*

2. *$v(\cdot, 0)$ has modulus of continuity φ_0, where $\varphi_0 : [0, D/2] \to \mathbb{R}$ is smooth with $\varphi_0(0) = 0$ and $\varphi_0'(z) > 0$ for $0 \leq z \leq D/2$;*

3. *$\varphi : [0, D/2] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies*

   (a) *$\varphi(z, 0) = \varphi_0(z)$ for each $z \in [0, D/2]$;*

   (b) *$\frac{\partial \varphi}{\partial t} \geq \varphi'' + \omega\varphi'$ on $[0, D/2] \times \mathbb{R}_+$;*

   (c) *$\varphi' > 0$ on $[0, D/2] \times \mathbb{R}_+$;*

   (d) *$\varphi(0, t) \geq 0$ for each $t \geq 0$.*

Then $\varphi(\cdot, t)$ is a modulus of continuity for $v(\cdot, t)$ for each $t \geq 0$.

Let u_1 and u_0 be two smooth solutions to the parabolic Schrödinger equation

$$\frac{\partial u}{\partial t} = \Delta u - Vu \quad \text{on } \Omega \times \mathbb{R}_+; \tag{5}$$

$$u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+,$$

on a bounded strictly convex domain Ω with smooth boundary in $\mathbb{R}^n$, with u_1 positive on the interior of Ω. Let $v(x, t) = \frac{u_2(x,t)}{u_1(x,t)}$. Both u_1 and u_2 are smooth on $\bar{\Omega} \times [0, \infty)$, and u_1 has negative derivative in the normal direction ν. It follows that v extends to $\bar{\Omega}$ as a smooth function. By computing directly,

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{u_2}{u_1}\right) \\
&= \frac{\Delta u_2 - V u_2 - v(\Delta u_1 - V u_1)}{u_1} \\
&= \Delta v + 2\nabla \log u_1 \cdot \nabla v.
\end{aligned} \tag{6}
$$

Moreover, at any point in $\partial\Omega$ we have $\frac{\partial v}{\partial t}$ and Δv bounded, $\nabla u_1 = -c\nu$ with $c > 0$, and $u_1 = 0$, so it follows from (6) that $D_\nu v = 0$ and then v satisfies (4) with $X = 2\nabla \log u_1$. Now, the goal is to use theorem 2.1 with $\varphi = C\frac{e^{-\mu_1 t}\tilde{\phi_2}}{e^{-\mu_1 t}\tilde{\phi_1}}$. Firstly, thanks to inequality 2, $X = 2\nabla \log u_1$ has modulus of contraction $\omega = 2(\ln \tilde{\phi}_1)'$ (condition 1 of theorem 2.1). Secondly, condition 2 holds for sufficiently large C, since $v(\cdot, 0)$ is smooth and so has a bounded gradient and satisfies $|v(y,0) - v(x,0)| \leq K|y - x|$ for some large K, while $\frac{\tilde{\phi_2}}{\tilde{\phi_1}}$ is positive on $(0, D(\Omega)/2)$ and has a positive gradient at $z = 0$, so it is bounded below by cz for small $c > 0$. Thirdly, the conditions 3 (i)-(iv) also hold, except that $\varphi'(D(\Omega)/2) = 0$ and ω is not smooth at $D(\Omega)/2$ since $\tilde{\phi}_1$ vanishes at this point. We handle these complications below by making slight adjustments to φ and ω.

For small $\varepsilon > 0$ we construct a smooth function $\omega_\varepsilon \geq \omega$ and corresponding functions $\varphi_{\varepsilon,\tilde{\varepsilon}}$ as follows: We first replace the Dirichlet eigenfunction $\tilde{\phi}_1$ by a Robin eigenfunction $\tilde{\phi}_{1,\varepsilon}$ with eigenvalue $\widetilde{\lambda_{1,\varepsilon}} < \tilde{\lambda}_1$ defined by

$$
\begin{cases}
\left(\tilde{\phi}_{1,\tilde{\varepsilon}}\right)'' - \tilde{V}\,\tilde{\phi}_{1,\tilde{\varepsilon}} + \widetilde{\lambda_{1,\varepsilon}}\tilde{\phi}_{1,\tilde{\varepsilon}} = 0, & \text{on } [0, D/2], \\
\tilde{\phi}_{1,\tilde{\varepsilon}}(D/2) = \tilde{\varepsilon}, \quad \tilde{\phi}'_{1,\tilde{\varepsilon}}(D/2) = -1, & \\
\tilde{\phi}'_{1,\tilde{\varepsilon}}(0) = 0, \quad \tilde{\phi}_{1,\tilde{\varepsilon}} > 0 \quad \text{on } [0, D/2]. &
\end{cases} \tag{7}
$$

We also replace $\tilde{\phi}_2$ by $\tilde{\phi}_{2,\tilde{\varepsilon}}$ with eigenvalue $\widetilde{\lambda_{2,\varepsilon}}$ defined by

$$
\begin{cases}
\left(\tilde{\phi}_{2,\tilde{\varepsilon}}\right)'' - \tilde{V}\,\tilde{\phi}_{2,\tilde{\varepsilon}} + \widetilde{\lambda_{2,\varepsilon}}\tilde{\phi}_{2,\tilde{\varepsilon}} = 0, & \text{on } [0, D/2], \\
\tilde{\phi}_{2,\tilde{\varepsilon}}(D/2) = \tilde{\varepsilon}, \quad \tilde{\phi}'_{2,\tilde{\varepsilon}}(D/2) = -1, & \\
\tilde{\phi}'_{2,\tilde{\varepsilon}}(0) = 0, \quad \tilde{\phi}_{2,\tilde{\varepsilon}} > 0 \quad \text{on } [0, D/2]. &
\end{cases} \tag{8}
$$

For any $\tilde{\varepsilon} > \varepsilon > 0$, by posing

$$
\varphi_{\varepsilon,\tilde{\varepsilon}} = \frac{e^{-\widetilde{\lambda_{2,\varepsilon}}t}\phi_{2,\varepsilon}}{e^{-\widetilde{\lambda_{1,\varepsilon}}t}\phi_{1,\varepsilon}} \text{ and } \omega_\varepsilon = 2(\ln \tilde{\phi}_{1,\varepsilon})',
$$

One can show that ω_ε is smooth, is a modulus of contraction of X (since $\omega_\varepsilon \geq \omega$) and $C\varphi_{\varepsilon,\tilde{\varepsilon}}(\cdot, t)$ is a modulus of continuity for v for $t > 0$ if it is so at $t = 0$. Now, since $\varphi_{\varepsilon,\tilde{\varepsilon}}$ converges uniformly to φ when $\tilde{\varepsilon} \to \varepsilon \to 0$, then the same conclusion holds for φ. In particular, for sufficiently large C we have

$$
\operatorname{osc} v(\cdot, t) \leq 2C \sup\{\varphi(z,t) : z \in [0, D/2]\} = 2Ce^{-(\mu_1 - \mu_0)t}.
$$

Applying this with $u_1 = \phi_1 e^{-\lambda_1 t}$ gives $v = e^{-(\lambda_1 - \lambda_0)t}\frac{\phi_1}{\phi_0}$ and hence

$$
e^{-(\lambda_1 - \lambda_0)t} \operatorname{osc} \frac{\phi_1}{\phi_0} \leq 2Ce^{-(\mu_1 - \mu_0)t}
$$

for all $t \geq 0$, which implies $\lambda_2(\Omega) - \lambda_1(\Omega) \geq \tilde{\lambda}_2 - \tilde{\lambda}_1$. $\square$ $\square$

Remark.

- *As said before, for the sphere, the same method is used, the inequality 1 becomes*

$$\langle \nabla V(y), \gamma'(\frac{d}{2})\rangle - \langle \nabla V(x), \gamma'(-\frac{d}{2})\rangle \leq 2\tilde{V}'\left(\frac{d(x,y)}{2}\right),$$

where $\gamma : [-\frac{d}{2}, \frac{d}{2}]$ represents the unique unit-speed length minimizing geodesic connecting x to y.

On the other hand, the super log-concavity estimates 2 becomes

$$\langle \nabla \log \phi_1(y), \gamma'\left(\tfrac{d}{2}\right)\rangle - \langle \nabla \log \phi_1(x), \gamma'\left(-\tfrac{d}{2}\right\rangle \leq -2\frac{\pi}{D}\tan\left(\frac{\pi d}{2D}\right) + (n-1)\,\tan\left(\tfrac{d}{2}\right).$$

- *One should emphasize that the super log-concavity estimate was necessary to obtain such a result, and it is substantially stronger than the log-concavity estimate that is typically used : a bound on $Hess(\log u_1)$. Indeed, let us assume that $Hess(\log u_1) \leq -C$ for a constant $C \in \mathbb{R}_+$. By posing $\omega = \frac{u_2}{u_1}$, where u_1, u_2 are are the first and second eigenfunctions of the Laplacian on Ω, then*

$$\Delta\omega + 2\langle \nabla \log u_1, \nabla\omega\rangle = -(\lambda_2(\Omega) - \lambda_1(\Omega)),$$

with Neumann boundary condition on $\partial\Omega$. By considering $\mathbb{S}^n$ with with weighted measure $u_1^2 dvol = e^{-(-2\log u_1)}dvol$, the Bakry-Emery Ricci curvature of this metric-measure space is $n-1-2Hess(\log u_1) \geq n-1+2C$. Thanks to [13] and [14] , the first nonzero Neumann eigenvalue of a weighted Laplacian is $\geq \dfrac{\pi^2}{D(\Omega)^2} + \dfrac{L}{2}$ with L a lower bound of the Bakry-Emery Ricci curvature. Then we deduce that

$$\lambda_2(\Omega) - \lambda_1(\Omega) \geq \frac{\pi^2}{D(\Omega)^2} + \frac{n-1}{2} + C.$$

Moreover, one can show that the super log-concavity estimates gives in particular the following inequality [12] :

$$Hess(\log u_1) \leq -(\frac{\pi^2}{D(\Omega)^2} - \frac{n-1}{2}K).$$

3 Stability of Γ_δ under Gromov-Hausdorff convergence

3.1 Gromov-Hausdorff convergence

Let's first investigate the weak and popular notion of *Gromov–Hausdorff convergence.* By definition, a family of compact metric spaces $(\mathcal{X}_k, d_k)_{k\in\mathbb{N}}$ converges to some metric space $(\mathcal{X}, d)$ in the *Gromov-Hausdorff* sense if there are (Borel measurable) maps $f_k : \mathcal{X}_k \to \mathcal{X}$, called *approximate isometries*, such that

$$\begin{cases} \forall x, y \in \mathcal{X}_k, & |d(f_k(x), f_k(y)) - d_k(x,y)| \leq \varepsilon_k, \\ \forall y \in \mathcal{X}, \exists x \in \mathcal{X}_k; & d(f_k(x), y) \leq \varepsilon_k, \end{cases}$$

where $(\varepsilon_k)_{k\in\mathbb{N}}$ is a sequence converging to 0. Moreover, if $\mathcal{X}_k$ (resp. $\mathcal{X}$) is equipped with reference Borel measures μ_k (resp. μ), we say that $(\mathcal{X}_k, d_k, \mu_k)_{k\in\mathbb{N}}$ converges to (X, d, μ) in the *measured Gromov-Hausdorff sense* if $(f_k)_{\#}\mu_k$ converges in the weak topology to μ. Note that there is no reason for f_k to be continuous maps. Nevertheless, one can still construct *approximate inverses* $f_k^{'} : \mathcal{X} \longrightarrow \mathcal{X}_k$ such that $d(f_k \circ f_k'(y), y) \leq \varepsilon_k$ and $d_k(f_k^{'} \circ f_k(x), x) \leq 2\varepsilon_k$, for all $(x, y) \in \mathcal{X}_k \times \mathcal{X}$ (see [5, section 1]). Since the Gromov-Hausdorff convergence takes care only about the distance, a direct implication of this convergence is this small and very useful lemma :

Lemma 3.1. *Let γ_k be a minimizing geodesic in $\mathcal{X}_k$ for each $k \in \mathcal{X}_k$. Then, up to extraction $f_k \circ \gamma_k$ converges (uniformly) to a minimizing geodesic in $\mathcal{X}$.*

Proof. Analogous proof in [3, Lemma 3.2]. □

Although this notion does not preserve the local structure, some geometric properties are stable under this convergence. For example, when $\mathcal{X}_k = M_k$ ($\mathcal{X} = M$) is a Riemannian manifold and all sectional curvatures of (M_k, g_k) are bounded from below by a fixed number $K \in \mathbb{R}$, then also the sectional curvatures of (M, g) are bounded from below by K. As mentioned before, in [4, section 4], one of the questions asked is whether the Γ_δ property is stable under this convergence of Riemannian manifolds. In this section, we prove a weaker version of the result, when each (M_k, g_k) satisfies two additional properties, which we define shortly. Let's first talk about the convex hull of a subset :

About the convex hull. Let Q be a subset of a Riemannian manifold M. The convex hull of a set $Q \subset M$ is the minimal convex subset of M that contains Q. Set $\mathfrak{C}^0(Q) = Q$ and let inductively $\mathfrak{C}^{i+1}(Q)$ be the union of all minimizing geodesics between pairs of points of $\mathfrak{C}^i(Q)$. By definition, the increasing countable union $\mathfrak{C}(Q) = \bigcup_i \mathfrak{C}^i(Q)$ is the convex hull of Q. Note that if M is complete and Q is compact, then each $\mathfrak{C}^i(Q)$ is compact. In the Euclidean space $M = \mathbb{R}^n$ (as well as in the round sphere), one can show that $\mathfrak{C}(Q) = \mathfrak{C}^n(Q)$. We are now ready to define two relevant properties that contribute to the stability of the Γ_δ property.

3.2 Caratheodory and GCE Properties

Definition 3.2. *A manifold (M^n, g) is said to satisfy the* ***Caratheodory*** *property if for each subset $Q \subset M^n$, we have $\mathfrak{C}(Q) = \mathfrak{C}^n(Q)$.*

Definition 3.3. *A manifold (M, g) is said to satisfy the* ***GCE*** *property (geodesic control from endpoints) if there exists a constant $C > 0$ such that: for any two minimizing geodesics (with constant speed) $\gamma_1, \gamma_2 : I \to M$,*

$$d(\gamma_1(t), \gamma_2(t)) \leq C \max\{d(\gamma_1(0), \gamma_2(0)), d(\gamma_1(1), \gamma_2(1))\}$$

for all $t \in I$.

Remark. *In particular, such a space requires the uniqueness of the minimizing geodesics.*

Example ($U_a = \{(x, y, z) \in \mathbb{S}^2, z > a\}$)**.** *For two points $p, q \in \mathbb{S}^2$ such that $p \neq -q$, the minimizing (with constant speed) geodesic between p and q is of the form*

$$\gamma_{p,q}(t) = \frac{\sin((1-t)\theta)}{\sin(\theta)} p + \frac{\sin(t\theta)}{\sin(\theta)} q, \ t \in [0, 1],$$

with $\theta = \arccos(\langle p, q\rangle)$. *By an easy computation we have*

$$\begin{cases} \dot\gamma_{p,q}(t) = \dfrac{-\theta\cos((1-t)\theta)}{\sin(\theta)}p + \dfrac{\theta\cos(t\theta)}{\sin(\theta)}q \\ \ddot\gamma_{p,q}(t) = -\theta^2\gamma_{p,q}(t). \end{cases}$$

Let $U_a = \{(x,y,z) \in \mathbb{S}^2, z > a\}$ (*with* $a \in]0,1[$). *Note that* U_a *is a convex domain of* $\mathbb{S}^2$. *Now, let* $\gamma_0, \gamma_1 : [0,1] \longrightarrow U_a$ *be two minimizing geodesics and let's take* α *(resp.* β*) the minimizing geodesic joining the points* $\gamma_0(0)$ *and* $\gamma_1(0)$ *(resp.* $\gamma_0(1)$ *and* $\gamma_1(1)$*).*
Thus by setting $\theta_s = \arccos(\langle\alpha(s), \beta(s)\rangle)$ *we define the following smooth function*

$$\Gamma(s,t) = \frac{\sin((1-t)\theta_s)}{\sin\theta_s}\alpha(s) + \frac{\sin(t\theta_s)}{\sin\theta_s}\beta(s)$$

which satisfies the following properties

- $\Gamma(0,\cdot) = \gamma_0, \Gamma(1,\cdot) = \gamma_1$
- $\Gamma(\cdot,0) = \alpha, \Gamma(\cdot,1) = \beta$
- *For all* $s \in [0,1]$, $t \mapsto \Gamma(s,t)$ *is the minimizing geodesic joining* $\alpha(s)$ *and* $\beta(s)$.

By construction, along each geodesic $\gamma_s := \Gamma(s,\cdot)$, *the vector field* $J_s := \partial_s\Gamma(s,\cdot)$ *is a Jacobi field :*

$$\frac{D^2}{dt^2}J_s(t) + R(J_s(t), \dot\gamma_s(t))\dot\gamma_s(t) = 0.$$

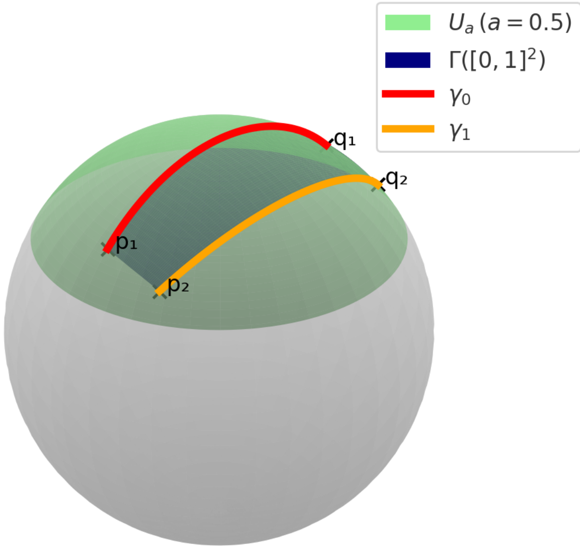
By posing $f_s(t) = \|J_s(t)\|^2 + \|\partial_t J_s(t)\|^2$, *we can easily check that* $\partial_t f_s \leq 2f_s$ ($\partial_t f_s \leq (1+C)f_s$ *in general if* $|sec| \leq C$) *and thus have the estimate* $\|J_s(t)\| \leq f_s(t) \leq f_s(0)e^{2t} = \|J_s(0) + \partial_t J_s(0)\|e^{2t}$. *Thanks to elementary estimates, we can explicitly give an upper bound in terms of* $d(\gamma_0(0), \gamma_1(0))$ *and* $d(\gamma_0(1), \gamma_1(1))$*:*

$$\|J_s(0) + \partial_t J_s(0)\|e^{2t} \leq C(a)\Big[d(\gamma_0(0), \gamma_1(0)) + d(\gamma_0(1), \gamma_1(1))\Big].$$

with a constant $C(a)$ *approaching* $+\infty$ *as* $a \to 0$.
Finally, we deduce for $t \in [0,1]$,

$$\boxed{d(\gamma_0(t), \gamma_1(t)) \leq \mathit{length}(s \mapsto \Gamma(s,t)) \leq \int_0^1 \|\partial_s\Gamma(s,t)\|ds \leq C(a)\Big[d(\gamma_0(0), \gamma_1(0)) + d(\gamma_0(1), \gamma_1(1))\Big]}$$

and thus U_a *satisfies the* ***GCE*** *property.*



Explicit illustration of γ_0, γ_1 *with* $a = 0.5$,

$$p_1 = \left(\frac{\sqrt{2}}{2},\ 0,\ \frac{\sqrt{2}}{2}\right),\ q_1 = \left(-\frac{\sqrt{2}}{2},\ 0,\ \frac{\sqrt{2}}{2}\right),$$

$$p_2 = \left(\frac{3\sqrt{2}}{\sqrt{38}},\ \frac{2}{\sqrt{38}},\ \frac{4}{\sqrt{38}}\right),\ q_2 = \left(-\frac{3\sqrt{2}}{\sqrt{38}},\ \frac{2}{\sqrt{38}},\ \frac{4}{\sqrt{38}}\right).$$

3.3 Convergence of convex subsets

Lemma 3.4. *let* $(M_k^n, g_k, dvol_k)$ *a sequence of compact Riemannian manifolds converging to* $(M^n, g, dvol)$ *in the measured Gromov–Hausdorff sense. Let's denote* $f_k : M_k^n \longrightarrow M^n$ *an approximate isometry. Let's assume that for all* k*,*

- (M_k^n, g_k) *satisfies the* ***GCE*** *property for a constant* $C > 0$ *independant of* k*.*
- (M_k^n, g_k) *satisfies the* ***Caratheodory*** *property.*

Then, for any convex domain $\Omega \subset M^n$*, the sequence* $(\mathfrak{C}(\Omega_k), g_k, dvol_k\big|_{\mathfrak{C}(\Omega_k)})$ *converges to* $(\Omega, g, dvol\big|_{\Omega})$ *in the measured Gromov-Hausdorff sense, where* $\Omega_k = f_k^{-1}(\Omega)$*.*

Proof. Let Ω be a convex domain of M^n and let us denote $\Omega_k = f_k^{-1}(\Omega)$, $\mu_k = dvol_k$, and $\mu = dvol$.

$$\textbf{Notation :}\ \text{For a subset } A \subset M^n,\ A^{(\delta)} := \{x \in M^n, d(x, A) \le \delta\}.$$

Claim. *There exists a sequence* $(\delta_k)_k$ *converging to* 0 *such that* $\mathfrak{C}^1(\Omega_k) \subset \Omega_k^{(\delta_k)}$*.*

Proof of the claim. let

$$\delta_k = \sup_{\gamma_k \in \mathfrak{C}^1(\Omega_k)} \sup_{t \in [0,1]} d_k(\gamma_k(t), \Omega_k).$$

Let us assume that $\limsup_k \delta_k > 0$, then up to extraction there exists $\eta > 0$, $x_k, y_k \in \Omega_k$, $\gamma_k : [0,1] \longrightarrow M_k$ a minimizing geodesic starting at x_k and ending at y_k such that

$\sup_{t\in[0,1]} d_k(\gamma_k(t),\Omega_k) \geq \eta,$ for all k. Up to extraction, let's denote $\tilde{x},\tilde{y}\in\Omega$ the respective limit of $f_k(x_k)$ and $f_k(y_k)$, and $\tilde{\gamma}:[0,1]\longrightarrow\Omega$ a minimizing geodesic between $\tilde{x}$ and $\tilde{y}$. Let $\alpha_k = f_k^{'}\circ\tilde{\gamma}$ (Recall that $f_k^{'}: M^n \longrightarrow M_k^n$ is an approximate inverse of f_k). By assumption, we show that $\sup_{t\in[0,1]} d_k(\gamma_k(t),\alpha_k(t)) \geq \eta - 4\varepsilon_k$. And by definition of f_k, we deduce that $\sup_{t\in[0,1]} d(f_k\circ\gamma_k(t),\tilde{\gamma}(t)) \geq \eta - 6\varepsilon_k$, which contradicts Lemma 3.1 and thus $\mathfrak{C}^1(\Omega_k)\subset\Omega_k^{(\delta_k)}$ with $\delta_k\longrightarrow 0$.

□

Next, let $\gamma_1:[0,1]\longrightarrow M_k^n$ be a minimizing geodesic between pairs of points of $\mathfrak{C}^1(\Omega_k)$, then from above there exists $a_k, b_k\in\Omega_k$ such that

$$d_k(a_k,\gamma_1(0))\leq\delta_k \quad\text{and}\quad d_k(b_k,\gamma_1(1))\leq\delta_k.$$

By taking $\gamma_2:[0,1]\longrightarrow M_k^n$ a minimizing geodesic between a_k and b_k, since (M_k,g_k) satisfies the **GCE** property for the constant C, we deduce that

$$\sup_{t\in[0,1]} d(\gamma_1(t),\gamma_2(t))\leq C\delta_k$$

and thus

$$\mathfrak{C}^2(\Omega_k)\subset\Omega_k^{((1+C)\delta_k)}.$$

Continuing this process, and since (M_k,g_k) satisfies the Carathéodory property, we deduce that

$$\Omega_k\subset\mathfrak{C}(\Omega_k)\subset\Omega_k^{(C'\delta_k)}$$

with C' another constant depending on the dimension and C.

Claim. $\mu_k\left(\mathfrak{C}(\Omega_k)\setminus\Omega_k\right)\xrightarrow[k\to\infty]{}0$. *Indeed, by the inclusion above, we have* $f_k(\mathfrak{C}(\Omega_k))\subset\Omega^{\beta_k}$ *for some* β_k *converging to* 0 *and then* $\mathfrak{C}(\Omega_k)\subset f_k^{-1}(\Omega^{\beta_k})$. *Thus, one can show that* $\mu_k(f_k^{-1}(\Omega^{\beta_k})) = (f_k)_{\#}\mu_k(\Omega^{\beta_k})\xrightarrow[k\to\infty]{}\mu(\Omega)$ *which proves the claim.*

Let $\tilde{f}_k:\mathfrak{C}(\Omega_k)\longrightarrow\Omega$ be constructed by sending a point $x\in\Omega_k$ to $f_k(x)$ and sending a point $x\in\mathfrak{C}(\Omega_k)\setminus\Omega_k$ to the projection of $f_k(x)$ to the boundary of Ω. One can show that $\tilde{f}_k$ is an approximate isometry for another sequence $\tilde{\varepsilon_k}$ converging to 0. Next, let $\varphi\in C_b(\Omega)$, by construction, we have

$$\int_{\Omega_k}\varphi\circ\tilde{f}_k\,du_k = \int_{\Omega_k}\varphi\circ f_k\,du_k\xrightarrow[k\to\infty]{}\int_{\Omega}\varphi\,d\mu,$$

where the convergence is due to the Portmanteau theorem. On the other hand, thanks to the last claim,

$$\int_{\mathfrak{C}(\Omega_k)}\varphi\circ\tilde{f}_k\,d\mu_k \;-\; \int_{\Omega_k}\varphi\circ\tilde{f}_k\,d\mu_k\xrightarrow[k\to\infty]{}0.$$

We can deduce that $(\tilde{f}_k)_{\#}(\mu_k\big|_{\mathfrak{C}(\Omega_k)})\rightharpoonup\mu\big|_{\Omega}$ and then that the sequence $(\mathfrak{C}(\Omega_k),g_k,\mu_k\big|_{\mathfrak{C}(\Omega_k)})$ converges to $(\Omega,g,\mu\big|_{\Omega})$ in the measured Gromov-Hausdorff sense. □

3.4 Spectrum convergence

Proposition 3.5. *Let $(M_k^n, g_k, dvol_k)$ and $(M^n, g, dvol)$ as the previous lemma. If, in addition, (M_k^n, g_k) satisfies Γ_δ for some $\delta > 0$ and for all k, then (M^n, g) also satisfies Γ_δ.*

Sketch of the proof. By keeping the same notation of the previous lemma, we know that

$$(\mathfrak{C}(\Omega_k), g_k, \mu_k\big|_{\mathfrak{C}(\Omega_k)}) \to (\Omega, g, \mu\big|_\Omega).$$

Obviously we have $diam(\mathfrak{C}(\Omega_k)) \xrightarrow[k\to\infty]{} diam(\Omega)$. It remains to prove the convergence of the eigenvalues $\lambda_i(\mathfrak{C}(\Omega_k)) \xrightarrow[k\to\infty]{} \lambda_i(\Omega)$. In [2, section 5, p.496], the author proved the spectral convergence for balls B_R with a fixed radius $R > 0$. Actually, we can use this result and adapt the proof in our case. We will denote the important estimates allowing the convergence of eigenvalues.

Let's take $R > 0$ sufficiently small such that for each k, $\mathfrak{C}(\Omega_k)$ contains a ball $B_R(x_k)$ of radius R with $x_k \in \Omega_k$, and let us denote u_k (resp. $\tilde{u}_k$) the eigenfunction solving the Dirichlet problems on $\mathfrak{C}(\Omega_k)$ (resp. $B_R(x_k)$) associated to the $i-$th eigenvalue $\lambda_i(\mathfrak{C}(\Omega_k))$ (resp. $\lambda_i(B_R(x_i))$).

Firstly, since $\mathrm{Ric}_k \geq 0$ and the domain $\mathfrak{C}(\Omega_k)$ is convex, we have the following gradient estimate (see [1]) :

$$\|\nabla u_k\|_\infty \leq A\|u_k\|_\infty\sqrt{\lambda_k(\mathfrak{C}(\Omega_k))} \;\text{ with } A = \sqrt{e}\left(\frac{\sqrt{2}}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{4\sqrt{2}}\right).$$

Secondly, since the sequence $(\lambda_i(B_R(x_k)))_k$ is bounded (see [2, Lemma 5.11]) and, by the decreasing monotonicity of Dirichlet eigenvalues under domain inclusion, so too is the sequence $(\lambda_i(\mathfrak{C}(\Omega_k)))_k$, which thus converges (up to extraction) to a limit λ_i^∞.

And thirdly, since we can see now that $\sup_k \|\nabla u_k\|_\infty < \infty$, by an analogous result of the Arzelà–Ascoli theorem, up to extraction, there exists a function u_∞ defined on Ω such that $u_k \xrightarrow[k\to\infty]{} u_\infty$ uniformly. Thanks to [2, Lemma 3.17], by replacing the balls by $\mathfrak{C}(\Omega_k)$ and by using a lower semicontinuity argument of energy via the convergence of the measures (see [2, Lemma 3.3]), this function solves the Dirichlet problem on Ω for the $i-$th eigenvalue. This completes the proof. □

Remark. *The results in Lemma 3.4 and in proposition 3.5 may hold when (M_k^n, g_k) does not verify the Carathéodory property, but with a weaker assumption. Indeed, in the proof of Lemma 3.4, the goal was to control the Hausdorff distance between Ω_k and $\mathfrak{C}(\Omega_k)$. If we suppose that there exists a uniform integer l such that*

$$\lim_{k\to+\infty} \sup_{Q\subset M_k^n} d_H^{(k)}(\mathfrak{C}^l(Q), \mathfrak{C}(Q)) = 0,$$

where $d_H^{(k)}$ is the Hausdorff distance in M_k^n, then the results remain true since we make small the Hausdorff distance between $\mathfrak{C}^l(\Omega_k)$ and $\mathfrak{C}(\Omega_k)$, and then control the distance between Ω_k and $\mathfrak{C}^l(\Omega_k)$ as in the proof in Lemma 3.4.

4 Lower-semicontinuous conditions

4.1 Uniformly Equivalent Metrics

Lemma 4.1. *Let g, g' two Riemannian metrics on M^n verifying $\alpha g \leq g' \leq \beta g$. Then,*

1. *For any smooth function f, the following inequality holds:*

$$\beta^{-1}\|\nabla^g f\|_g^2 \leq \|\nabla^{g'} f\|_{g'}^2 \leq \alpha^{-1}\|\nabla^g f\|_g^2.$$

2. *The volume forms are related by:*

$$\alpha^{n/2}\, d\mathrm{Vol}_g \leq d\mathrm{Vol}_{g'} \leq \beta^{n/2}\, d\mathrm{Vol}_g.$$

3. *For any domain $\Omega \subset M^n$,*

$$\sqrt{\alpha}D(g,\Omega) \leq D(g',\Omega) \leq \sqrt{\beta}D(g,\Omega),$$

where $D(g,\Omega)$ is the diameter of Ω for the metric g.

4. *For any smooth domain $\Omega \subset M^n$ and for any integer $k \geq 1$,*

$$\frac{\alpha^{n/2}}{\beta^{1+n/2}}\lambda_k(g,\Omega) \leq \lambda_k(g',\Omega) \leq \frac{\beta^{n/2}}{\alpha^{1+n/2}}\lambda_k(g,\Omega),$$

where $\lambda_k(g,\Omega)$ is the k-th eigenvalue of the laplacian Δ_g with Dirichlet conditions on the boundary of Ω.

Proof. .

1. **Control of the gradient term:** By definition of the gradient, we have for any smooth function $f \in C^\infty(M^n)$ and for any vector field $X \in \Gamma(M^n)$,

$$df(X) = g(\nabla^g f, X) = g'(\nabla^{g'} f, X).$$

By taking $X = \nabla^g f$, we have

$$\|\nabla^g f\|_g^2 = df(\nabla^g f) = g'(\nabla^{g'} f, \nabla^g f) \leq \|\nabla^{g'} f\|_{g'} \cdot \|\nabla^g f\|_{g'} \leq \sqrt{\beta}\|\nabla^{g'} f\|_{g'} \cdot \|\nabla^g f\|_g$$

and thus $\|\nabla^g f\|_g \leq \sqrt{\beta}\|\nabla^{g'} f\|_{g'}$.

On the other hand, by starting from $\|\nabla^{g'} f\|_{g'}^2$ with the same method, we obtain the other inequality.

2. **Comparison of volume forms:** Let $p \in M^n$ and $e_1, ..., e_n$ be an orthonormal basis of T_pM^n for the metric g. We denote $g_{i,j} = g(e_i, e_j) = \delta_{i,j}$ and $g'_{i,j} = g'(e_i, e_j)$. Since $(g'_{i,j})_{i,j}$ is symmetric positive definite, up to a change of basis, we can write

$$(g'_{i,j})_{i,j} = diag(\lambda_1, ..., \lambda_n),$$

with $0 < \lambda_1 \leq ... \leq \lambda_n$.
By assumption, we have for all $v \in T_pM^n$,

$$\alpha\|v\|_g^2 \leq \|v\|_{g'}^2 \leq \beta\|v\|_g^2.$$

By taking $v = e_i$ we have

$$\alpha \leq \lambda_i \leq \beta$$

and thus

$$\alpha^{n/2} \leq \sqrt{\lambda_1 \cdots \lambda_n} \leq \beta^{n/2}.$$

Finally, by writing in local coordinates in p,

$$dV_g(x) = \sqrt{det(g_{i,j}(x))}dx_1 \wedge ... \wedge dx_n \text{ and } dV_g'(x) = \sqrt{det(g'_{i,j}(x))}dx_1 \wedge ... \wedge dx_n,$$

we obtain $dV_{g'}(p) = \sqrt{\lambda_1 \cdots \lambda_n}dV_g(p)$ and thus

$$\alpha^{n/2}dV_g(p) \leq dV_{g'}(p) \leq \beta^{n/2}dV_g(p).$$

3. **Comparison of diameter :** immediate by definition of the diameter.
4. **Comparison of eigenvalues :** Let $\Omega \subset M^n$ a smooth domain and $f \in H_0^1(\Omega)$. By the above results, we have

$$\int_\Omega \|\nabla^{g'} f\|_{g'}^2 dV_{g'} \leq \alpha^{-1}\beta^{n/2} \int_\Omega \|\nabla^g f\|_g^2 dV_g$$

and

$$\int_\Omega f^2 dV_{g'} \geq \alpha^{n/2} \int_\Omega f^2 dV_g.$$

Thus we deduce that

$$R_{g'}[f] = \frac{\int_\Omega \|\nabla^{g'} f\|_{g'}^2 dV_{g'}}{\int_\Omega f^2 dV_{g'}} \leq \frac{\beta^{n/2}}{\alpha^{1+n/2}} \frac{\int_\Omega \|\nabla^g f\|_g^2 dV_g}{\int_\Omega f^2 dV_g} = \frac{\beta^{n/2}}{\alpha^{1+n/2}} R_g[f].$$

Finally, by min-max principle, we deduce the result.

□

4.2 CTL property

Let us denote $I(M^n, g) = \inf_{\Omega \in Conv(M^n,g)}(\lambda_2(g,\Omega) - \lambda_1(g,\Omega))D(g,\Omega)^2$.

Definition 4.2. *We say that (M^n, g) verifies the **CTL** property if*

$\forall\ \varepsilon > 0, \exists\ A = A(g,\varepsilon) > 0, \forall\ \Omega \in Conv(M^n, g)$,

$$(\lambda_2(g,\Omega) - \lambda_1(g,\Omega))D(g,\Omega)^2 \leq I(M^n, g) + \varepsilon \implies \lambda_1(g,\Omega)D(g,\Omega)^2 \leq A.$$

Proposition 4.3. *Let (M^n, g_0) be a bounded Riemannian manifold. Let $(g(t))_{t\in[0,\tau)}$ be a C^1-in-time family of Riemannian metrics with $g(0) = g_0$. Let us assume that for all $t \in [0, \tau)$,*

- *$(M^n, g(t))$ satisfies the **CTL** property, for all $t \in [0,\tau)$,*
- *There exists $\varepsilon > 0$ such that $C = \sup_{t\in[0,\tau)} A(g(t),\varepsilon) < \infty$, with $A(t,\varepsilon) = A(g(t),\varepsilon)$ the constant given in the definition 4.2 for the metric $g(t)$.*
- *$Conv(M^n, g(t)) \subset Conv(M, g_0)$, for all $t \in [0,\tau)$.*

Then the map

$$H : t \in [0,\tau) \longmapsto \inf_{\Omega\in Conv(M,g(t))} (\lambda_2(t,\Omega) - \lambda_1(t,\Omega))D(t,\Omega)^2 \in \mathbb{R}$$

is lower semicontinuous at 0.

As noted in the introduction, these properties are purely theoretical; they provide sufficient conditions for proposition 4.3. It is currently unknown whether a space satisfying them exists. For the **CTL** property, the key point is to control the first eigenvalue when the gap approaches the lower bound wanted; in [4], a lower bound of the fundamental gap has been found for each convex domain $\Omega \subset \mathbb{S}^n$ in the round sphere,

$$\lambda_2(\Omega) - \lambda_1(\Omega) > \frac{\pi^2}{D(\Omega)^2} + \frac{1}{2}(\sqrt{n^2 + 4\lambda_1(\Omega)} - 1). \tag{9}$$

On the other hand, for manifolds with Ricci curvature bounded below by $(n-1)K$, we have the following lower bound for the first eigenvalue [10] :

$$\lambda_1(\Omega) > \frac{(n-1)K}{2} + \frac{\pi^2}{4r(\Omega)^2} \tag{10}$$

where $r(\Omega) := \sup_{x\in\Omega} d(x, \partial\Omega)$ is the radius of the largest interior ball in Ω. Combining this with inequality 9, we obtain for the sphere

$$(\lambda_2(\Omega) - \lambda_1(\Omega))D(\Omega)^2 > D(\Omega)^2 \frac{1}{2}\left(\sqrt{\frac{\pi^2}{r(\Omega)^2}} - 1\right) = D(\Omega)^2\left(\frac{\pi}{2\,r(\Omega)} - \frac{1}{2}\right).$$

This give an upper bound of the quantity $\dfrac{D(\Omega)^2}{r(\Omega)}$. Then, we can deduce that to approach $I(\mathbb{S}^n, g_{\mathbb{S}^n})$, the domain must not be too thin if the diameter is bounded from below. But what about the quantity $\dfrac{D(\Omega)}{r(\Omega)}$? Can we obtain an upper bound? In any case, if a manifold (with a strict positive lower bound for the Ricci curvature) satisfies the **CTL** property, this must be a necessary condition by the inequality 10, and thus the domain Ω will be necessarily sufficiently round and not too thin.

Corollary 4.4. *Under the same assumptions of the above proposition, if (M^n, g_0) satisfies the Γ_δ property for some $\delta > 0$, then for all $\delta' < \delta$, there exists $T > 0$ such that for all $t \in [0,T]$, $(M^n, g(t))$ satisfies the $\Gamma_{\delta'}$ property.*

Proof of the proposition 4.3. Let $\varepsilon > 0$ given by the second assumption. Let us denote

$$Gap(t,\Omega) = \lambda_2(t,\Omega) - \lambda_1(t,\Omega).$$

Without loss of generality, we can assume that $K = \sup_{t\in[0,\tau)} \|\partial_t g(t)\|_{g(t)} < \infty$. By Gronwall's lemma, we have for all $s, t \in [0, \tau)$,

$$e^{-K|t-s|}g(s) \leq g(t) \leq e^{K|t-s|}g(s).$$

By lemma 4.1, we deduce that for any smooth domain $\Omega \subset M^n$,

$$e^{-(n+1)K|t-s|}\lambda_k(s,\Omega) \leq \lambda_k(t,\Omega) \leq e^{(n+1)K|t-s|}\lambda_k(s,\Omega)$$

and thus

$$|Gap(t,\Omega) - Gap(s,\Omega)| \leq (2(n+1)K|t-s| + o(|t-s|))(Gap(s,\Omega) + 2\lambda_1(s,\Omega)) \quad (11)$$

with the approximation $e^{(n+1)K|t-s|} = 1 + (n+1)K|t-s| + o(|t-s|)$.
Next, since $(M^n, g(t))$ satisfies the **CTL** property and since $Conv(M^n, g(t)) \subset Conv(M^n, g_0)$, we have for all $t \in [0, \tau)$,

$$\begin{aligned} I(M^n, g(t)) &= \inf_{\Omega \in Conv(M^n,g(t)) \ | \ \lambda_1(t,\Omega)D^2(t,\Omega)\leq C} Gap(t,\Omega)D^2(t,\Omega) && (12) \\ &\geq \inf_{\Omega\in\mathcal{F}} Gap(t,\Omega)D^2(t,\Omega) && (13) \end{aligned}$$

with $\mathcal{F} = \{\Omega \in Conv(M^n, g_0) \mid \lambda_1(0,\Omega)D^2(0,\Omega) \leq e^{(n+1)K\tau}C\}$

Claim. *We can restrict $\mathcal{F}$ to $\mathcal{F}' = \{\Omega \in \mathcal{F},\ Gap(0,\Omega)D(0,\Omega)^2 \leq I(M, g_0) + \varepsilon\}$*

on a small interval $[0, T_1) \subset [0, \tau)$. For the instance, let's assume the claim is true.
Thanks to the inequality 11 and again the lemma 4.1, we can deduce that on a small interval $[0, T_2) \subset [0, T_1)$, we have the following inequality for any $\Omega \in \mathcal{F}$,

$$|Gap(t,\Omega) - Gap(s,\Omega)| \leq C'|t-s|(Gap(0,\Omega) + e^{(n+1)K\tau} \cdot C \cdot D(0,\Omega)^{-2} \quad (14)$$

and

$$|D(t,\Omega)^2 - D(s,\Omega)^2| \leq C'|t-s|D(0,\Omega)^2,$$

with C' depending on n,K and τ only.
Finally, for any $\Omega \in \mathcal{F}'$, from the above results, the map

$$f_\Omega : t \in [0, T_2) \longmapsto Gap(t,\Omega)D(t,\Omega)^2 \in \mathbb{R}$$

is a Lipschitz map for a constant independent of the domain Ω and thus the map

$$F : t \in [0, T_2) \longmapsto \inf_{\Omega\in\mathcal{F}'} Gap(t,\Omega)D^2(t,\Omega) \in \mathbb{R}$$

is continuous and the result follows since $H(t) = I(M^n, g(t)) \geq F(t)$ with $F(0) = I(M^n, g_0)$.

Proof of the claim. Indeed, suppose the contrary, that there exists a sequence $(t_k)_k$ converging to 0 such that for all k,

$$\inf_{\Omega\in\mathcal{F}'} Gap(t_k,\Omega)D^2(t_k,\Omega) > \inf_{\Omega\in\mathcal{F}} Gap(t_k,\Omega)D^2(t_k,\Omega).$$

It follows that there exists $\Omega_k \in \mathcal{F} \setminus \mathcal{F}'$ so that

$$\inf_{\Omega \in \mathcal{F}'} Gap(t_k, \Omega) D^2(t_k, \Omega) > Gap(t_k, \Omega_k) D(t_k, \Omega_k)^2.$$

But the left member can be smaller than $I(M^n, g_0) + \varepsilon/2$ for k large enough, and furthermore from the inequality 11, we have

$$Gap(0, \Omega_k) D(0, \Omega_k)^2 - Gap(t_k, \Omega_k) D(t_k, \Omega_k)^2 \xrightarrow[k \to \infty]{} 0$$

with $Gap(0, \Omega_k) D(0, \Omega_k)^2 \geq I(M^n, g_0) + \varepsilon$.
We obtain a contraction for k large enough. $\square$

$\square$

References

[1] M.Arnaudon, A.Thalmaier, F-Y.Wang, *Gradient Estimates on Dirichlet and Neumann Eigenfunctions*, International Mathematics Research Notices, 2018.

[2] Yu Ding, *Heat kernels and Green's functions on limit spaces*, Analysis and geometry, 2002.

[3] C.Villani, *Stability of a 4th-order curvature condition arising in optimal transport theory*, Journal of Functional Analysis, 2008.

[4] G.Khan, X. H. Nguyen, M. Tuerkoenn, G.Wei, *Log-Concavity and Fundamental Gaps on Surfaces of Positive Curvature*, Communications in Analysis and Geometry 33, 2025.

[5] C.Villani, J.Lott, *Ricci curvature for metric-measure spaces via optimal transport*, Annals of Mathematics, 2009.

[6] I. M. Singer, B. Wong, S.-T. Yau, and S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa, 1985.

[7] Q. H. Yu and J. Q. Zhong, *Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator* Trans. Amer. Math. Soc. 294 (1986), no. 1, 341–349.

[8] J. Ling, *A lower bound for the gap between the first two eigenvalues of Schrödinger operators on convex domains in* $\mathbb{S}^n$ or $\mathbb{R}^n$, Michigan Math. J. **40** (1993), no. 2, 259-270.

[9] B. Andrews and J. Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. **24** (2011), no. 3, 899–916.

[10] Ling, Jun. *A lower bound of the first Dirichlet eigenvalue of a compact manifold.* International Journal of Mathematics, 17(5):605–617, 2006.

[11] G.Khan and X.H. Nguyen. *Negative curvature constricts the fundamental gap of convex domains.* Annales Henri Poincaré 25 (11), 4855-4887, 2024.

[12] X. Dai, S. Seto, and G. Wei. *Fundamental gap estimate for convex domains on sphere—the case n=2.* Comm. in Analysis and Geometry 29, No. 5, 1095-1125, 2021.

[13] Andrews, Ben; Ni, Lei. *Eigenvalue comparison on Bakry-Emery manifolds.* Communications in Partial Differential Equations, 37(11):2081–2092, 2012.

[14] Bakry, Dominique; Qian, Zhongmin. *Some new results on eigenvectors via dimension, diameter, and Ricci curvature.* Advances in Mathematics, 155(1):98–153, 2000.