Non-uniqueness of Turbulent Solutions of the Navier-Stokes Equation in Dimension $N = 3$

Thomas Giarrizzi, Yuran Wang

February 15, 2025

Abstract

In this article, we review the conditional uniqueness property for Navier-Stokes equation, and follow the paper [ABC22] to develop a non-uniqueness theorem of 3-dimensional case with source term. We study the linearized operator of the self-similar equation, then reduce it gradually to a 2-dimensional Euler equation. The construction of the stationary unstable velocity field relies fundamentally on Vishik's unstable profile in [Vis18a] and [Vis18b], which we then lift into 3-dimensional axisymmetric coordinate.

The main improve compared to the original paper is that we composed the proofs in an uniform way, viewing the linearized operators of perturbations of skew-adjoint operators. The case without exterior force is still an open problem.

1 Introduction

NOTE: Due to the length request of the Memoire, this article is a short version which only contains sketch of proof. If you are interested in the full-length proof, please refer to the longer version.

In this article, we are interested in the following **Navier-Stokes Equation** System with source term in dimension $N = 2, 3$, which describes the evolution of local velocity with respect to time of a viscous non-compressible fluid.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u = f - \nabla p, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ \nabla \cdot u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u|_{t=0} = u_0, & x \in \mathbb{R}^N \end{cases} \tag{N-S}$$

where $u : \mathbb{R}_{\geq 0} \times \mathbb{R}^N \to \mathbb{R}^N$ is a N-dimensional vector field, $p : \mathbb{R}_{\geq 0} \times \mathbb{R}^N \to \mathbb{R}$ is the pressure function, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^N \to \mathbb{R}^N$ is the given external force applied to the fluid, and $u_0 : \mathbb{R}^N \to \mathbb{R}^N$ is the given initial velocity field at time $t = 0$. The parameter $\nu > 0$ is called the viscosity factor, illustrates how much resistance the fluid produces when being deformed at a given rate, the term

$-\nu\Delta$ gives the regularity of the system. When there is no viscosity, i.e. $\nu = 0$, then we obtain the **Euler equation** with source term:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = f - \nabla p, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ \nabla \cdot u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u|_{t=0} = u_0, & x \in \mathbb{R}^N \end{cases} \tag{Euler}$$

In the light of the proper rescaling below, we assume that $\nu \equiv 1$:

$$u'(t, x) := \nu u(\nu t, x), \quad p'(t, x) := \nu^2 p(\nu t, x), \quad f'(t, x) := \nu^2 f(\nu t, x)$$

Since u is divergence free, we apply the divergence operator to the first equation of system (N-S) and obtain the dependence of p on u and f:

$$\Delta p = \nabla \cdot f - \nabla \cdot ((u \cdot \nabla) u) \tag{1.1}$$

Here we have to introduce some notations, using row vectors, we could write the velocity and divergence operator as $u = (u_1, \cdots, u_N)$, $\nabla = (\partial_1, \cdots, \partial_N)$, we then define the operations below:

1 For two vectors $u^{(\alpha)} = (u_1^{(\alpha)}, u_2^{(\alpha)}, \cdots, u_N^{(\alpha)}) \in \mathbb{R}^N$, $\alpha = 1, 2$, we define the tensor product of these two vectors as a $N \times N$ matrix :

$$u^{(1)} \otimes u^{(2)} := (u^{(1)})^T (u^{(2)}) = (u_i^{(1)} u_j^{(2)})_{1 \le i,j \le N} \in \mathbb{R}^{N \times N}$$

2 For two $N \times N$ matrices $A^{(\alpha)} = (A_{ij}^{(\alpha)})_{1 \le i,j \le N} \in \mathbb{R}^{N \times N}$, $\alpha = 1, 2$, we define the contraction product of them:

$$A^{(1)} : A^{(1)} := \sum_{1 \le i,j \le N} A_{ij}^{(1)} A_{ij}^{(2)}$$

3 For a matrix $A = (A_{ij})_{1 \le i,j \le N} \in \mathbb{R}^{N \times N}$, we define the divergence of A:

$$\nabla \cdot A = (\partial_1, \cdots, \partial_N) A^T = \left(\sum_{j=1}^N \partial_j A_{ij} \right)_{1 \le i \le N}$$

Then for all $u, v \in C^\infty(\mathbb{R}^N)^N$ such that $\nabla \cdot v = 0$, we have:

$$\nabla \cdot (u \otimes v) = (u \cdot \nabla) v + u (\nabla \cdot v) = (u \cdot \nabla) v \tag{1.2}$$

For $u \in C_w([0, T], L^2(\mathbb{R}^N)) \cap L^2([0, T], \dot{H}^1(\mathbb{R}^N))$, where C_w refers to continuous with respect to the weak topology, define the kinetic energy:

$$E_u(t) := \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \nu \|u\|_{L^2([0,t], \dot{H}^1(\mathbb{R}^N))}^2, \quad \forall\, t \in [0, T] \tag{1.3}$$

For $T > 0$, we define the proper space $\mathcal{G}_T$ of the external force:

$$\mathcal{G}_T := L^1([0, T], L^2(\mathbb{R}^N)^N) + L^2([0, T], \dot{H}^{-1}(\mathbb{R}^N)^N)$$

We follow the approach of Jean Leray [Ler34], and define the solution:

Definition 1.1 (Turbulent Solution)**.** *For all times $T > 0$, let $u_0 \in L^2(\mathbb{R}^N)^N$, such that $\nabla \cdot u_0 = 0$, $f \in \mathcal{G}_T$, we say that $u \in \mathcal{S}'([0,T] \times \mathbb{R}^N)^N$ is a turbulent solution of equation (N-S) on the time interval $[0,T]$, if:*

$$u \in C_w([0,T], L^2(\mathbb{R}^N)^N) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^N)^N)$$

For all function $\psi \in \mathcal{D}([0,T] \times \mathbb{R}^N)$, for all $t \in [0,T]$, we have:

$$\int_{\mathbb{R}^N} \nabla\psi(t,x) \cdot u(t,x) \mathrm{d}x = 0$$

And for all functions $\phi \in C^1([0,T], H^1(\mathbb{R}^N)^N)$,such that $\nabla \cdot \phi = 0$, we have:

$$\begin{aligned} &\int_{\mathbb{R}^N} u(T,x) \cdot \phi(T,x) \mathrm{d}x - \int_0^T \int_{\mathbb{R}^N} u(s,x) \cdot \partial_t \phi(s,x) \mathrm{d}x \mathrm{d}s \\ &\quad = -\nu \int_0^T \int_{\mathbb{R}^N} \nabla u(s,x) : \nabla \phi(s,x) \mathrm{d}x \mathrm{d}s + \int_{\mathbb{R}^N} u_0(x) \cdot \phi(0,x) \mathrm{d}x \\ &\quad\quad + \int_0^T \int_{\mathbb{R}^N} u \otimes u(s,x) : \nabla\phi(s,x) \mathrm{d}x \mathrm{d}s + \int_0^T \int_{\mathbb{R}^N} f(s,x) \cdot \phi(s,x) \mathrm{d}x \mathrm{d}s \end{aligned} \tag{1.4}$$

Finally, the solution satisfies the following Energy Inequality:

$$E_u(t) \leq \frac{1}{2} \|u_0\|^2_{L^2(\mathbb{R}^N)} + \int_0^t \int_{\mathbb{R}^N} f \cdot u \ \mathrm{d}x \mathrm{d}s \qquad \forall\ t \in [0,T] \tag{1.5}$$

[Ler34] has provided the existence of turbulent solution:

Proposition 1.1. *Let $N = 2,3$, given initial data $u_0 \in L^2(\mathbb{R}^N)^N$ divergence free, then for all $T > 0$, if $f \in \mathcal{G}_T$ there exists a turbulent solution u of the system (N-S) on time interval $[0,T]$.*

For the uniqueness of the turbulent solution, [LP59] has proved it for dimension $N = 2$, which essentially relies on the fact that proper rescaling does not change the corresponding $L^\infty_t L^2_x$ and $L^2_t \dot{H}^1_x$ norm of the solution.

The problem for the uniqueness for general turbulent solutions in dimension $N = 3$ has remained long open. Leray's work has proven the "strong-weak" uniqueness of the turbulent solution, i.e. if the solution is strong, which satisfies a certain regularity condition, then all turbulent solutions coincide.

In this article, we consider the equation (N-S) with source term not regular enough to admit a strong solution, then, the "strong-weak" principle cannot be applied. We follow the lines of [ABC22], to construct a source term which gives a counter-example for the uniqueness in dimension $N = 3$:

Theorem 1.1. *Assume $\nu = 1$, then there exists a time $T > 0$, an external source term $f \in L^1([0,T], L^2(\mathbb{R}^N)^N)$, and two distinct turbulent solutions u and $\tilde{u}$ to the system (N-S) on time interval $[0,T]$ with initial condition $u_0 \equiv 0$.*

2 Leray's Turbulent Solution

This section is dedicated to give an outline of the proof of Leray's existence theorem (1.1), we consider the case $f \in L^2_{loc}(\mathbb{R}, \dot{H}^{-1}(\mathbb{R}^N)^N)$, the general case could be treated with minor modification.

In this section, we use the following convention:

$$\mathcal{H} := L^2(\mathbb{R}^N)^N, \quad \mathcal{V} := H^1(\mathbb{R}^N)^N, \quad \mathcal{V}^{'} := \dot{H}^{-1}(\mathbb{R}^N)^N$$

and the space with subscript σ is the divergence-free subspace.

2.1 Construction of Approximate Solutions

Firstly, we want to construct a sequence of approximate solutions to the Navier-Stokes equation.

To get rid of the pressure p, we apply the following Leray operator onto the equation (N-S):

$$\begin{aligned} \mathbf{P}: \quad & \mathcal{M} && \to && \mathcal{M} \\ & f = (f_1, \cdots, f_N) && \mapsto && \left(\delta_{i,j} - \frac{D_i D_j}{|D|^2}\right)_{i,j=1}^N f \end{aligned} \tag{2.1}$$

where $\mathcal{M} = H^k(\mathbb{R}^N)^N$ or $\dot{H}^k(\mathbb{R}^N)^N$, $k \in \mathbb{R}$. We use the Fourier multiplier convention:

$$\left(\delta_{i,j} - \frac{D_i D_j}{|D|^2}\right)_{i,j=1}^N f = \mathcal{F}^{-1}\left[\left(\delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2}\right)_{i,j=1}^N \hat{f}\right]$$

Let $(\mathbf{P}_k)_{k\in\mathbb{N}}$ be a family of spectral projections approximating $\mathbf{P}$, which satisfies the following properties : $\forall f \in \mathcal{V}^{'}$, then we have $\mathbf{P}_k f \in \mathcal{V}_\sigma$, and:

$$\|\mathbf{P}_k f\|_{\mathcal{V}'_\sigma} \leq \|f\|_{\mathcal{V}'_\sigma}, \quad \lim_{k\to+\infty} \|\mathbf{P}_k f - \mathbf{P} f\|_{\mathcal{V}'_\sigma} = 0$$

Due to the length request, we deliberately omit the construction of $(\mathbf{P}_k)_{k\in\mathbb{N}}$, denote by $\mathcal{H}_k$ the space $\mathbf{P}_k\mathcal{H}$.

Definition 2.1. *Let us define the (continuous) bilinear map*

$$\mathcal{Q}: \begin{cases} \mathcal{V}\times\mathcal{V} & \longrightarrow \quad \mathcal{V}^{'} \\ (u,v) & \longmapsto \quad -\nabla\cdot(u\otimes v) = -\left(\sum_{i=1}^N \partial_i(u_i v_j)\right)_{1\leq j\leq N} \end{cases}.$$

The continuity of $\mathcal{Q}$ from the Sobolev embedding. Moreover, using integration by parts and a smoothing argument, we have:

$$\langle \mathcal{Q}(u,v), v\rangle = 0, \quad (u,v) \in \mathcal{V}_\sigma \times \mathcal{V}$$

Let us take $(f_k)_{k\in\mathbb{N}}$ an approximation of f in the space $\mathcal{C}^1(\mathbb{R}_+,\mathcal{V}_\sigma)$ such that $f_k(t)\in\mathcal{H}_k$ and $\lim_{k\to\infty}\|f_k-f\|_{L^2([0,T],\mathcal{V}'_\sigma)}=0$. Now, let us introduce the following ordinary differential equation :

$$\begin{cases} \dot{u}_k(t)=\mathbf{P}_k\Delta u_k(t)+F_k(u_k(t))+f_k(t) \\ u_k(0)=\mathbf{P}_k u_0 \end{cases} \tag{2.2}$$

with $F_k(u)=\mathbf{P}_k\mathcal{Q}(u,u)$.

1. Thanks to linearity of $\mathbf{P}_k\Delta:\mathcal{H}_k\longrightarrow\mathcal{H}_k$ and continuity of $\mathcal{Q}$, the Cauchy–Lipschitz theorem give the existence of a unique maximal solution u_k of (NS_k) in $\mathcal{C}^\infty([0,T_k[,\mathcal{H}_k)$ with $T_k\in]0,+\infty]$.

2. We have a control of $\|\dot{u}_k(t)\|_{L^2}$ with the following inequality : $\|\dot{u}_k(t)\|_{L^2}\leq k\|u_k(t)\|_{L^2}+Ck^{\frac{d}{4}}\|u_k(t)\|^2_{L^2}+\|f_k(t)\|_{L^2}$

3. If $T_k<+\infty$ and $\|u_k(t)\|_{L^2}$ is bounded on the interval $[0,T_k[$, so does $\|\dot{u}_k(t)\|_{L^2}$ and then we can extend the solution beyond T_k.

4. Actually, because each u_k verifies the energy equality, we can show that $(u_k)_{k\in\mathbb{N}}$ remains uniformly bounded on $\mathcal{H}$ for all time, hence that $T_k=+\infty$. Moreover, $(u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^\infty_{loc}(\mathbb{R}_+,\mathcal{H})\cap L^2_{loc}(\mathbb{R}_+,\mathcal{V}_\sigma)\cap L^{\frac{8}{d}}_{loc}(L^4(\Omega))$ and $(\Delta u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^2_{loc}(\mathbb{R}_+,\mathcal{V}'_\sigma)$.

2.2 Compactness Properties

The next result will be crucial for the rest of the proof. It will give us a candidate for the global weak solution to (N-S).

Proposition 2.1. *A vector field u exists in $L^2_{loc}(\mathbb{R}_+,\mathcal{V}_\sigma)$ such that up to an extraction (which we omit to note) $(u_k)_{k\in\mathbb{N}}$ converges to u strongly in $L^2_{loc}(\mathbb{R}_+\times\mathbb{R}^N)$ and weakly in $L^2_{loc}(\mathbb{R}_+,\mathcal{V}_\sigma)$. Moreover, $(u_k(t))_{k\in\mathbb{N}}$ converges weakly to $u(t)$ uniformly on $[0,T]$ for functions in $C^1(\mathbb{R}_+,\mathcal{V}_\sigma)$.*

Let us consider a test function $\Psi\in C^1(\mathbb{R}_+,\mathcal{V}_\sigma)$. By definition of u_k, we have

$$\begin{aligned}\frac{d}{dt}\langle u_k(t),\Psi(t)\rangle =&\langle\dot{u}_k(t),\Psi(t)\rangle+\langle u_k(t),\dot{\Psi}(t)\rangle\\ =&\langle P_k\Delta u_k(t),\Psi(t)\rangle+\langle P_k\mathcal{Q}(u_k(t),u_k(t)),\Psi(t)\rangle\\ &+\langle f_k(t),\Psi(t)\rangle+\langle u_k(t),\dot{\Psi}(t)\rangle.\end{aligned}$$

After integrating by parts and integrating in time from 0 to t, we obtain

$$\int_\Omega u_k(t,x)\cdot\Psi(t,x)dx+\int_0^t\int_\Omega(\nabla u_k:\nabla P_k\Psi-u_k\otimes u_k:\nabla P_k\Psi-u_k\cdot\partial_t\Psi)(t',x)dxdt'$$

$$=\int_\Omega u_k(0,x)\cdot\Psi(0,x)dx+\int_0^t\langle f_k(t'),\Psi(t')\rangle dt'.$$

By using the local strong convergence of $(u_k)_{k\in\mathbb{N}}$ we can pass to the limit and prove that u is a solution of (N-S) in the sense of (1.1).

2.3 End of the Proof

Thanks to Proposition 1.2, we can show by using properties of weak convergence that for all $t \geq 0$,

$$\|u(t)\|_{L^2}^2 \leq \liminf_{k\to+\infty} \|u_k(t)\|_{L^2}^2\,, \qquad \int_0^t \|\nabla u(t')\|_{L^2}^2\, dt' \leq \liminf_{k\to+\infty} \int_0^t \|\nabla u_k(t')\|_{L^2}^2\, dt'.$$

Then we can deduce the energy inequality.

Remark : This is precisely at this moment that we lose the energy equality. Actually, we can show that u_k verifies also the energy equality, but the weak convergence of $(u_k)_k$ does not allow u to verify the equality, reason why we have only an inequality and then we cannot maintain the uniqueness of the solutions.

Finally, let's prove the time continuity of u : for any $\Psi \in \mathcal{V}_\sigma$, because u satisfies the weak formulation,

$$\langle u(t), \Psi\rangle - \langle u(t'), \Psi\rangle = \int_t^{t'} \langle f(s), \Psi\rangle ds + \int_t^{t'} \int_\Omega (\nabla u(s) : \nabla\Psi - u\otimes u(s) : \nabla\Psi) dx ds.$$

The inequality $\|u(t)\|_{L^4} \leq \|u(t)\|_{L^2}^{1-\frac{d}{4}} \|\nabla u(t)\|_{L^2}^{\frac{d}{4}}$ implies $u \in L^{\frac{8}{d}}([0,T], L^4(\mathbb{R}^N)^N)$. Then we deduce that

$$\begin{aligned} |\langle u(t), \Psi\rangle - \langle u(t'), \Psi\rangle| \leq & |t-t'|^{1-\frac{d}{4}} \|u\|_{L^{\frac{8}{d}}([0,T],L^4(\Omega))}^2 \|\Psi\|_{\mathcal{V}_\sigma} + \\ & |t-t'|^{\frac{1}{2}} (\|f\|_{L^2([0,T],\mathcal{V}'_\sigma)} + \|\nabla u\|_{L^2([0,T],L^2)}) \|\Psi\|_{\mathcal{V}_\sigma}\,. \end{aligned}$$

which concludes the proof of (1.1).

3 Previous Results: A Revisit

In this section, we review the uniqueness properties of the turbulent solutions in dimension $N = 2, 3$.

3.1 Case 2D: Unconditional Uniqueness

Proposition 3.1. *Given initial data* $u_0 \in L^2(\mathbb{R}^2)^2$, $f \in L^1([0,T], \dot{H}^{-1}(\mathbb{R}^2)^2)$, *then there exists a unique turbulent solution* u *of the system (N-S) defined on time interval* $[0,T]$.

Sketch of Proof. The proof of uniqueness essentially relies on Sobolev embedding and Grönwall's inequality. Assume that there exists two turbulent solution (u, p_u) and (v, p_v), where p denotes the associated pressure function.

Apply the Leray projector (2.1) onto the equation, we obtain:

$$\begin{cases} \partial_t w - \Delta w = -\mathbf{P}(\nabla\cdot(u\otimes u - v\otimes v)) \\ \nabla\cdot w = 0 \\ w|_{t=0} \equiv 0 \end{cases} \tag{3.1}$$

Apply Sobolev inequality $H^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$,

$$\|\mathbf{P}(\nabla \cdot (u \otimes u - v \otimes v))\|_{L^2_t \dot{H}^{-1}_x} \lesssim (\|u\|_{L^4_t \dot{H}^{\frac{1}{2}}_x} + \|v\|_{L^4_t \dot{H}^{\frac{1}{2}}_x})\|w\|_{L^4_t \dot{H}^{\frac{1}{2}}_x}$$

And by interpolation inequality,

$$\|f\|_{L^4_t \dot{H}^{\frac{1}{2}}_x} \lesssim \|f\|^{\frac{1}{2}}_{L^\infty([0,T],L^2(\mathbb{R}^2)^2)} \|f\|^{\frac{1}{2}}_{L^2([0,T],\dot{H}^1(\mathbb{R}^2)^2)}$$

Thus, the source term is in the classical energy space, apply standard energy estimate for heat equation, combine all estimates above:

$$\begin{aligned}
&\frac{1}{2}\|w(t)\|^2_{L^2(\mathbb{R}^2)^2} + \int_0^t \|w(s)\|^2_{\dot{H}^1(\mathbb{R}^2)^2} \mathrm{d}s \\
&\lesssim \int_0^t (\|u(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2} + \|v(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2}) \|w(s)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)^2} \|w(s)\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^2)^2} \mathrm{d}s \\
&\leq \int_0^t c(\|u(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2} + \|v(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2})^4 \|w(s)\|^2_{L^2(\mathbb{R}^2)^2} + \frac{1}{2}\|w(s)\|^2_{\dot{H}^1(\mathbb{R}^2)^2} \mathrm{d}s
\end{aligned}$$

where in the last line we applied the multi-variable case for arithmetic-geometric inequality. Absorb $\mathcal{H}^1$ term in the right, and the conclusion follows from Grönwall's inequality. ☐

3.2 Case 3D: Strong-Weak Principle

Proposition 3.2. *Given initial data $u_0 \in L^2(\mathbb{R}^3)^3$, $f \in L^2([0,T], \dot{H}^{-1}(\mathbb{R}^3)^3)$, then if the system (N-S) admits a turbulent solution u defined on $[0,T]$, furthermore satisfies the extra regularity below:*

$$u \in L^4([0,T], \dot{H}^1(\mathbb{R}^3)^3)$$

Then, every turbulent solution v defined on $[0,T]$ coincide with u.

Sketch of Proof. Similarly, we consider the difference of two solutions:

$$w := u - v \in C_w([0,T], L^2(\mathbb{R}^3)^3) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^3)^3)$$

Then, we have the following estimate of energy comes from the energy inequality:

$$\begin{aligned}
E_w(t) \leq \|u_0\|_{L^2(\mathbb{R}^3)^3} + \int_0^t \int_{\mathbb{R}^N} f \cdot (u+v) \, \mathrm{d}x\mathrm{d}s \\
- \langle u(t), v(t)\rangle_{L^2(\mathbb{R}^3)^3} - 2\int_0^t \langle \nabla u(s), \nabla v(s)\rangle_{L^2(\mathbb{R}^3)^9} \mathrm{d}s
\end{aligned} \tag{3.2}$$

The extra regularity of u guarantees that we could couple the (N-S) of u with v and vice-versa:

$$\int_0^t \int_{\mathbb{R}^3} \partial_t u \cdot v + (\nabla u - u \otimes u) : \nabla v \, \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{R}^3} f \cdot v \, \mathrm{d}x\mathrm{d}s \tag{3.3}$$

$$\langle v(t), u(t)\rangle_{L^2(\mathbb{R}^3)^3} - \int_0^t \int_{\mathbb{R}^3} v \cdot \partial_t u + (v \otimes v - \nabla v) : \nabla u \, \mathrm{d}x\mathrm{d}s$$
$$= \|u_0\|^2_{L^2(\mathbb{R}^3)^3} + \int_0^t \int_{\mathbb{R}^3} f \cdot u \, \mathrm{d}x\mathrm{d}s \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we have:

$$E_w(t) \leq \int_0^t \int_{\mathbb{R}^3} u \otimes u : \nabla v + v \otimes v : \nabla u \, \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla)u \cdot w \, \mathrm{d}x\mathrm{d}s$$

The last equality is by divergence free property. Apply Sobolev embedding:

$$\left| \int_{\mathbb{R}^3} (w \cdot \nabla)u \cdot w \, \mathrm{d}x \right| \lesssim \|w\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)^3} \|w\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^3)^3} \|u\|_{\dot{H}^1(\mathbb{R}^3)^3}$$

Similarly absorb $\dot{H}^1$ term, and the conclusion follows from Grönwall's inequality.

□

4 Non-Uniqueness: Review and Foreseen

In this section, we are going to briefly review the history of non-uniqueness problem of Navier-Stokes equation, and give a self-similar refined version of the main theorem (1.1).

4.1 A Brief History

Non-Uniqueness of Navier-Stokes equation has been a long question in different contexts, less regular the solution is, the more non-uniqueness it may holds.

In the weakest sense, Buckmaster and Vicol [BV19] have proven that in the class of finite kinetic energy distributions, for any given initial datum, the solution is always non-unique. However, their solution lies in the finite energy class $L^\infty_t L^2_x$ but is not regular enough to be a Leray solution.

But in the context of Leray's turbulent solution, the problem remains open. The method adopted by [ABC22], which is presented in this paper, is partially inspired by Vishik's study of non-uniqueness of Euler's equation with external force in [Vis18a] and [Vis18b]. The main idea is to consider a unstable steady state, and apply an external force as perturbation in $L^1_t L^p_x$. We will utilize his method of constructing an **unstable vortex solution**.

How about removing the external force? It's still an open problem for the uniqueness of (N-S) without external force. But in the context of Euler equation, Bressan, Murray, and Shen have proposed a potential initial data which potentially leads to non-unique solutions without external force in [BM20] and [BS21]. We still need further exploration for similar conclusions on Navier-Stokes equation.

4.2 Strategy of Refining

The scheme is described below: The term **unstable** refers to the fact that the linearized operator around a given stationary profile in some equivalent coordinate has an unstable eigenvalue, and the corresponding perturbation term turns to 0 while $t \to -\infty$. It's hard to treat the question on positive time $\mathbb{R}^+$, thus, we introduce the **similarity variables**:

$$\xi := \frac{x}{\sqrt{t}}, \quad \tau := \log t \in \mathbb{R} \tag{4.1}$$

$$u(t,x) =: \frac{1}{\sqrt{t}} U(\tau, \xi), \quad f(t,x) =: \frac{1}{t^{\frac{3}{2}}} F(\tau, \xi) \tag{4.2}$$

Here the lower case letters denote the functions in physical space, and the upper case letters denote functions with similarity variables. The Navier-Stokes equation becomes:

$$\begin{cases} \partial_\tau U - \frac{1}{2}(1 + \xi \cdot \nabla_\xi) U - \Delta U + U \cdot \nabla U = F - \nabla P, & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot U = 0, & (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3 \end{cases} \tag{4.3}$$

We call a solution $\tilde{U}$ to (4.3) **linear unstable**, if the linearized operator $\mathbf{L}_{\mathrm{ss}}$ at $\tilde{U}$ defined below has eigenvalue with positive real part.

$$\mathbf{L}_{\mathrm{ss}}(U) := \frac{1}{2}(1 + \xi \cdot \nabla_\xi) U + \Delta U + \mathbb{P}(\tilde{U} \cdot \nabla U + U \cdot \nabla \tilde{U}) \tag{4.4}$$

where $\mathbb{P}$ is the Leray projector defined in (2.1). We will study the **unstable manifold** associated with the most unstable eigenvalue λ_0, i.e. the eigenvalue with the largest real part which we denote as $a = \Re(\lambda_0) \in \mathbb{R}$. Then, the solutions on this manifold satisfy the following asymptotic behavior:

$$U = \tilde{U} + U_{\mathrm{lin}} + O(e^{2\tau a}), \quad \tau \to -\infty$$

where U^{lin} is a non-trivial solution to the linearized equation $\partial_t U^{\mathrm{lin}} = \mathbf{L}_{\mathrm{ss}} U^{\mathrm{lin}}$ on $\mathbb{R} \times \mathbb{R}^3$ corresponding to an eigenfunction of λ_0. Since U^{lin} decays at the rate $e^{\tau a}$ as $\tau \to -\infty$, thus,

$$\lim_{\tau \to -\infty} U = \tilde{U}$$

Since we can view $\tau = -\infty$ as $t = 0$, and by the formula (4.2), the corresponding velocity field in physical space satisfies:

$$\lim_{t \to 0^+} \|u(t)\|_{L^2(\mathbb{R}^3)^3} = \lim_{\tau \to -\infty} e^{\frac{1}{2}\tau} \|U(\tau)\|_{L^2(\mathbb{R}^3)^3} = 0 \tag{4.5}$$

The same holds true for $\tilde{u}$, this corresponds to the non-uniqueness of $t = 0$ in time variable with initial value $u_0 = 0$. We obtain a refined version of (1.1):

Theorem 4.1. *There exists a smooth, compactly supported stationary real-valued velocity field $\tilde{U}$ which does not depend on the self-similar time τ, and a smooth, compactly supported source term*

$$\tilde{F} := -\frac{1}{2}(1 + \xi \cdot \nabla_\xi)\tilde{U} - \Delta\tilde{U} + \tilde{U} \cdot \nabla\tilde{U}$$

satisfying the following properties:

1 *The linearized operator $\mathbf{L}_{\text{ss}}$ defined by (4.4) has an unstable eigenvalue λ with a non-zero eigenfunction $\eta \in H^k(\mathbb{R}^3)$ for all $k \geq 0$. Denote one of the solutions to the linearized equation $\partial_t U^{\text{lin}} = \mathbf{L}_{\text{ss}} U^{\text{lin}}$ as below:*

$$U^{\text{lin}}(\tau) := \Re(e^{\lambda\tau}\eta)$$

2 *There exists a time-like upper bound $T \in \mathbb{R}$ and a velocity field U^{per}, satisfying the following smallness condition. For all $k \in \mathbb{N}$, we have:*

$$\|U^{\text{per}}(\tau)\|_{H^k(\mathbb{R}^3)} \lesssim_k e^{2\tau a}, \quad \forall \tau \in (-\infty, T]$$

And in view of (4.5), $\tilde{u}$ and $\tilde{u} + u^{\text{lin}} + u^{\text{per}}$ are two distinct turbulent solutions to the system (N-S) with initial value $u_0 \equiv 0$ and external force $\tilde{f}$.

Furthermore, we can make the following observations:

- *The external force gives us a large range of choice. And by simply modifying the amplitude, we can view nonlinear convection term as the leading term, and the other terms, including viscosity, act as small perturbations.*

Thus, the problem is reduced to finding a smooth unstable steady state for the 3-dimensional Euler equation with external force. It's natural to impose extra symmetry and lower the dimension to 2D case, a frequently considered family of steady state functions are so-called **vortices**, which write in polar coordinates as:

$$\bar{u}(r, \theta) = \bar{u}^\theta(r)e_\theta, \quad (r, \theta) \in \mathbb{R}_+ \times (\mathbb{R}\backslash 2\pi\mathbb{Z})$$

where e_θ denotes the clockwise tangent vector at $(1, \theta)$ of the unit circle. Vishik has constructed an unstable vortex with power-law decay as $r \to +\infty$. Our method is lifting Vishik's vortex in a axisymmetric without swirl way into 3 dimensions, of the form

$$u(r, \theta, z) = u^r(r, z)e_r + u^z(r, z)e_z$$

where $e_r,\ e_z$ are respectively the unit vector in the r and z-direction. The corresponding vorticity and stream function are of following form:

$$\omega := \nabla \times u = -\omega^\theta(r, z)e_\theta \quad \varphi = -\Delta^{-1}\omega = \varphi^\theta(r, z)e_\theta \tag{4.6}$$

Then, with these variables, applying the curl operator on the Euler equation (Euler) with $f = 0$, together with the definition of ω, φ, leads to the following system:

$$\begin{cases} \partial_t \omega^\theta + u \cdot \nabla \omega^\theta - \dfrac{u^r}{r}\omega^\theta = 0 \\ w^\theta = \left(\partial_r^2 + \dfrac{1}{r}\partial_r - \dfrac{1}{r^2}\right)\varphi^\theta \\ u = -\partial_z \varphi^\theta e_r + \left(\partial_r + \dfrac{1}{r}\right)\varphi^\theta e_z \end{cases} \tag{4.7}$$

As $r \to +\infty$, all the terms of order r^{-1} and r^{-2} can be neglected, so we reduce to the following 2-dimensional Euler-like system:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ w = \Delta \varphi, \quad u = \nabla^\perp \varphi \end{cases} \tag{4.8}$$

Hence, we can lift Vishik's unstable 2D vortex into a 3D **hollow vortex ring** with large diameter in order to ease the influence of those higher order terms. This suggests us a possible way to construct the unstable steady state.

5 Instability: From 2D to Axisymmetric Case

In this section, we study the instability property of the 3-dimensional axisymmetrical Euler system (4.8). It relies fundamentally on the construction of Vishik's unstable vortex for 2-dimensional Euler's equation in [Vis18a] and [Vis18b].

5.1 2D Instability: Vishik's Vortex

This section is devoted to stating a version of **unstable vortex** constructed by Vishik in [Vis18b].

We briefly state the **Biot–Savart Operator** to construct a velocity profile when given a vorticity:

$$\begin{aligned} \mathrm{BS}_{2d} : L^2(\mathbb{R}^2) &\quad\to\quad \dot{H}^1(\mathbb{R}^2)^2 \\ \omega &\quad\mapsto\quad \nabla \times \Delta^{-1}\omega \end{aligned} \tag{5.1}$$

and we can write the image above into the convolution form:

$$\nabla \times \Delta^{-1}\omega = K_{\mathrm{BS}_{2d}} * \omega, \quad K_{\mathrm{BS}_{2d}} = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \tag{5.2}$$

where for $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$.

Remark 5.1. *Following the same procedure, we can define the Biot–Savart operator* $\mathrm{BS}_{3d} : L^2(\mathbb{R}^3)^3 \to \dot{H}^1(\mathbb{R}^3)^3$ *in the 3-dimensional case:*

$$\mathrm{BS}_{3d}[\omega](x) = \int_{\mathbb{R}^3} K_{\mathrm{BS}_{3d}}(y) \times \omega(x-y)\mathrm{d}y, \quad K_{\mathrm{BS}_{3d}}(x) := -\frac{1}{4\pi}\frac{x}{|x|^3} \in L_w^{\frac{3}{2}}(\mathbb{R}^3) \tag{5.3}$$

Let us consider the behavior of the linearized operator associated with the Euler-like equation (4.8) around a steady state. Consider a smooth, real-valued, divergence-free vector field with sufficient decay at $r \to +\infty$, under the form:

$$\bar{u}(r,\theta) = \zeta(r)e_\theta$$

Set the corresponding vorticity which is a radial function:

$$\bar{\omega} = \nabla \times \bar{u} = \partial_r \zeta + \frac{1}{r}\zeta$$

We would like to work on a subspace of $L^2(\mathbb{R}^2)$ to obtain better decay properties of our eigenfunction. Let us introduce our desired weighted space, where we use the coordinate (x, y) in $\mathbb{R}^2$

$$L^2_\gamma := L^2(\mathbb{R}^2, \gamma \mathrm{d}x\mathrm{d}y) \tag{5.4}$$

where the smooth weight function $\gamma > 0$ satisfies:

$$\gamma|_{B_0(\bar{R})} \equiv 1, \quad (1 + x^2 + y^2)^{50} \lesssim \gamma \lesssim (1 + x^2 + y^2)^{50} \tag{5.5}$$

Now we consider the operator $\mathbf{L}_\infty$ defined below:

$$\begin{aligned} \mathbf{L}_\infty : \mathrm{D}(\mathbf{L}_\infty) \quad &\to \quad L^2_\gamma \\ \omega \quad &\mapsto -(u \cdot \nabla\bar{\omega} + \bar{u} \cdot \nabla\omega) \end{aligned} \tag{5.6}$$

where $u = \mathrm{BS}_{2d}[\omega]$, and the domain of definition

$$\mathrm{D}(\mathbf{L}_\infty) = \{\omega \in L^2_\gamma | \bar{u} \cdot \nabla\omega \in L^2_\gamma\}$$

.

In fact, $u \cdot \nabla\bar{w}$ is compact by Rellich's embedding if given $\nabla\tilde{\omega}$ with compact support, and $\bar{u} \cdot \nabla\omega$ is skew-adjoint. We will use the following scheme for several times:

Remark 5.2. *Consider a skew-adjoint linear operator* $\mathbf{L}$ *define on* $\mathcal{H}$*, we perturb* $\mathbf{L}$ *with a compact operator* $\mathbf{T}$*. Then, the essential spectrum satisfies:*

$$\sigma_{\mathrm{ess}}(\mathbf{L} + \mathbf{T}) \subset \{x \in \mathbb{C} | \Re(x) \le 0\}$$

and the remaining spectrum of $\mathbf{L} + \mathbf{T}$ *consists of isolated eigenvalues of finite order in* $\{x \in \mathbb{C} | \Re(x) > 0\}$*.*

We now state a modified version of Vishik's unstable vortex:

Theorem 5.1. *There exists a smooth, real, compactly supported velocity field expressed in polar coordinate* $\bar{u}(r,\theta) = \bar{\zeta}(r)e_\theta$*, with its corresponding smooth vorticity* $\bar{\omega}(r,\theta) = \bar{\omega}(r)$*, satisfying that the linearized operator* $\mathbf{L}_\infty$ *has an unstable eigenvalue* λ*, i.e.* $\Re(\lambda) > 0$*.*

5.2 Axisymmetric Instability

We want to lift the previous argument to the **3-dimensional axisymmetric-without-swirl case**, with coordinates written as $(r', z) \in \mathbb{R}_+ \times \mathbb{R}$, the function u defined on this coordinate is independent of the angle θ, and of the form:

$$u(r', z) = u^r(r', z)e_r + u^z(r', z)e_z, \quad \omega := \nabla \times u = -\omega^\theta(r, z)e_\theta$$

In the discussion below, we will **use 2D Euclidean coordinates to represent 3D behavior**, beware that the operators are different from the original 2D operators!

To ease the influence of r^{-1} and r^{-2} term, the lifted vortex should be away from 0. Thus, we work on the shifted variable $r := r' - l \in \mathbb{R}_{\geq -l}$, where $l > 2\bar{R}$.

From now on, we use the following convention:

Convention 1. *All the operators with subscript l are the 2-dimensional operators induced by the corresponding 3-dimensional operators applied to the shifted axisymmetric-without-swirl variable $(r', z) = (r + l, z)$.*

Denote our 2-dimensional unstable vortex in **Euclidean coordinate** $(r, z) \in \mathbb{R}^2$ in alignment with 3D axisymmetrical case as:

$$\bar{u}(r, z) = \bar{u}^r(r, z)e_r + \bar{u}^z(r, z)e_z$$

By the discussion above, it can be lifted to a 3-dimensional axisymmetric-without-swirl profile in the coordinate (r, z), denoted also by $\bar{u}$. Notice that $\bar{u}$ is not a 3D divergence-free profile in the shifted coordinate (r', z):

$$\nabla_l \cdot \bar{u} := \left(\partial_r + \frac{1}{r+l}\right)\bar{u}^r + \partial_z \bar{u}^z = \frac{1}{r+l}\bar{u}^r$$

where ∇_l is the operator induced by the gradient of 3D shifted cylindrical coordinate (r', z). Observe that the error term is decaying as $l \to \infty$, we have:

Lemma 5.1. *For all $l \geq 2\bar{R}$, there exists a real correction $v_l \in C_0^\infty(B_0(\bar{R}), \mathbb{R}^2)$, satisfying:*

$$\nabla_l \cdot (\bar{u} + v_l) = 0$$

and the following decay property:

$$v_l \underset{l\to\infty}{\longrightarrow} 0 \quad in \quad C^k(B_0(\bar{R})) \quad \forall k \geq 0$$

Sketch of Proof. Only need to solve the following equation:

$$\nabla_l \cdot u = \frac{1}{r+l}(\partial_r, \partial_z) \cdot [(r+l)u] \equiv (\partial_r, \partial_z) \cdot [(r+l)v_l] = -\bar{u}^r$$

Recalling that $\bar{u}^r \in C_0^\infty(B_0(\bar{R}))$ with $\int_{\mathbb{R}^2} \bar{u}^r(r, z)\mathrm{d}r\mathrm{d}z = 0$ inherited from its rotational symmetry, applying the Bogovskii's operator in case $N = 2$, we obtain the desired conclusion. $\square$

For any $l > 2\bar{R}$, we can define the corrected background profile:

$$\bar{u}_l := \bar{u} + v_l \in C_0^\infty(B_0(\bar{R}), \mathbb{R}^2), \quad \bar{\omega}_l := \nabla_l \times \bar{u}_l \in C_0^\infty(B_0(\bar{R}))$$

As the flow is axisymmetric without swirl, *we can identify the vorticity with its angular component as a scalar function:*

$$\bar{\omega}_l = -\bar{\omega}_l^\theta e_\theta, \quad \bar{\omega}^\theta = -\partial_z \bar{u}_l^r + \partial_r \bar{u}_l^z$$

We consider the linearized operator of equation (4.7) around the background profile $(\bar{u}_l, \bar{\omega}_l)$. Firstly, we have to define the proper functional space we work on. As before, we expect that the functions are decaying sufficiently fast:

$$L^2_{\gamma,l} := L^2(\mathbb{R}_{r\geq -l} \times \mathbb{R}_z, \gamma \mathrm{d}r\mathrm{d}z)$$

where γ is the same weight function defined in (5.5).

Define the linearized operator:

$$\begin{aligned} \mathbf{L}_l : \mathrm{D}(\mathbf{L}_l) \quad &\rightarrow \quad L^2_{\gamma,l} \\ \omega \quad &\mapsto \quad \mathbf{L}_l\omega \end{aligned} \tag{5.7}$$

where,

$$-\mathbf{L}_l\omega := \bar{u}_l \cdot \nabla\omega + \mathrm{BS}_l[\omega] \cdot \nabla\bar{\omega}_l - \frac{\bar{u}_l^r}{r+l}\omega - \frac{(\mathrm{BS}_l[\omega])^r}{r+l}\bar{\omega}_l \tag{5.8}$$

$\nabla = (\partial_r, \partial_z)$ is the normal gradient operator, BS_l is the operator induced by 3-dimensional Biot-Savart operator BS_{3d} applied to the axisymmetric coordinate $(r', z) = (r + l, z)$, satisfy the following system of equations:

$$\begin{aligned} \Delta_l\varphi_l := \left(\partial_r^2 + \frac{1}{r+l}\partial_r - \frac{1}{(r+l)^2} + \partial_z^2\right)\varphi_l = \omega \quad in \quad \mathbb{R}_{r\geq -l} \times \mathbb{R} \\ \partial_r\varphi_l|_{r=-l} = 0 \\ \mathrm{BS}_l[\omega] := -\partial_z\varphi_l e_r + \left(\partial_r + \frac{1}{r+l}\right)\varphi_l e_z \end{aligned} \tag{5.9}$$

The domain of definition is:

$$\mathrm{D}(\mathbf{L}_l) := \{\omega \in L^2_{\gamma,l} | \bar{u}_l \cdot \nabla\omega \in L^2_{\gamma,l}\}$$

We observe the following decomposition of the operator $\mathbf{L}_l$:

$$\mathbf{L}_l = \mathbf{M}_l + \mathbf{K}_l + \mathbf{S}_l$$

into the **main** skew-symmetric term, the **compact** term, and the **small** term:

$$-\mathbf{M}_l\omega := \bar{u}_l \cdot \nabla\omega + \frac{1}{2}\omega\nabla \cdot v_l$$

$$-\mathbf{K}_l\omega := \mathrm{BS}_l(\omega) \cdot \nabla\bar{\omega}_l$$

$$-\mathbf{S}_l\omega := -\frac{\bar{u}_l^r}{r+l}\omega - \frac{(\mathrm{BS}_l(\omega))^r}{r+l}\bar{\omega}_l - \frac{1}{2}\omega\nabla \cdot v_l$$

Note that the additional term $\frac{1}{2}\omega\nabla \cdot v_l$ is to make $\mathbf{M}_l$ a skew-adjoint operator.

Here are the basic properties of the compact and small operators:

Lemma 5.2. $\mathbf{K}_l, \mathbf{S}_l : L^2_{\gamma,l} \to L^2_{\gamma,l}$ *are two bounded operators. Moreover,* $\mathbf{K}_l$ *is a compact operator, and*

$$\|\mathbf{S}_l\|_{L^2_{\gamma,l} \to L^2_{\gamma,l}} \underset{l \to \infty}{\longrightarrow} 0$$

Sketch of Proof. The compactness of $\mathbf{K}_l$ essentially relies on Rellich's theorem, and the smallness of $\mathbf{S}_l$ is intuitive. ☐

Since we are considering l large enough, as the small term converges to zero, only need to consider the formal limits of the main and compact term:

$$\begin{array}{rcccc} \mathbf{K}_\infty : & L^2_\gamma & \to & L^2_\gamma \\ & \omega & \mapsto & -\mathrm{BS}_{2d}[\omega] \cdot \nabla \bar{\omega} \\ \mathbf{M}_\infty : \mathrm{D}(\mathbf{M}_\infty) \subset & L^2_{\gamma,l} & \to & L^2_{\gamma,l} \\ & \omega & \mapsto & -\bar{u} \cdot \nabla \omega \end{array}$$

where the domain of definition:

$$\mathrm{D}(\mathbf{M}_\infty) = \{\omega \in L^2_\gamma | \bar{u} \cdot \nabla \omega \in L^2(B_0(\bar{R}))\}$$

We observe that the total formal limit $\mathbf{M}_\infty + \mathbf{K}_\infty = \mathbf{L}_\infty$ has an unstable eigenvalue by the discussion of the previous subsection. Thus, we expect that for l large enough, the operator $\mathbf{L}_l$ is unstable:

Proposition 5.1. *Let* λ_∞ *be an unstable eigenvalue of* $\mathbf{L}_\infty$*, then, for all* $\epsilon \in (0, \Re(\lambda_\infty))$*, there exist* $l_0 = l_0(\bar{u}, \lambda_\infty, \epsilon) \geq 2\bar{R}$*, such that for all* $l \geq l_0$*, the operator* $\mathbf{L}_l$ *has an unstable eigenvalue* λ_l *with* $|\lambda_l - \lambda_\infty| < \epsilon$

The original problem can be transformed into the generalized lemma below, and we can justify all the conditions with the setting of proposition above.

Lemma 5.3. *Let* $(\mathbf{M}_n)_{n\in\mathbb{N}}$*,* $\mathbf{M}_\infty : \mathrm{D}(\mathbf{M}_\infty) \subset \mathcal{H} \to \mathcal{H}$ *closed normal operators,* $(\mathbf{K}_n)_{n\in\mathbb{N}}$*,* $\mathbf{K}_\infty : \mathcal{H} \to \mathcal{H}$ *compact operators,* $(\mathbf{S}_n)_{n\in\mathbb{N}} : \mathcal{H} \to \mathcal{H}$ *bounded operators, satisfying the following properties:*

1 There exist $(\mu_n)_{n\in\mathbb{N}\cup\{\infty\}} \in \mathbb{R}$*, such that* $\lim_{n\to\infty} \mu_n = \mu_\infty$*, and:*

$$\sigma(\mathbf{M}_n) \subset \{z \in \mathbb{C} | \Re(z) \leq \mu_n\}, \quad \forall n \in \mathbb{N} \cup \{\infty\}$$

2 For all $f \in \mathcal{H}$*,*

$$(\lambda - \mathbf{M}_n)^{-1} f \underset{n\to\infty}{\longrightarrow} (\lambda - \mathbf{M}_\infty)^{-1} f \quad \text{in } \mathcal{H} \tag{5.10}$$

locally uniformly on $\lambda \in \{z \in \mathbb{C} | \Re(z) > \mu_\infty\}$

3 $\mathbf{K}_n \longrightarrow \mathbf{K}_\infty$ *in operator norm.*

4 The smallness of $\{\mathbf{S}_n\}_{n\in\mathbb{N}}$*:*

$$\lim_{n\to\infty} \|\mathbf{S}_n\|_{\mathcal{H}\to\mathcal{H}} = 0 \tag{5.11}$$

Let λ_∞ be an isolated eigenvalue of $\mathbf{M}_\infty + \mathbf{K}_\infty$, such that $\Re(\lambda_\infty) > \mu_\infty$, and let $V \subset \{z \in \mathbb{C} | \Re(z) > \mu_\infty\}$ such that $V \cap \sigma(\mathbf{M}_\infty + \mathbf{K}_\infty) = \{\lambda_\infty\}$, then, for n big enough, there exist an eigenvalue of $\mathbf{M}_n + \mathbf{K}_n + \mathbf{S}_n$ in V.

Sketch of Proof. We claim that: for all $C \subset \operatorname{res}(\mathbf{C}_\infty) \cap \{z \in \mathbb{C} | \Re(z) > \mu_\infty\}$ compact, $f \in \mathcal{H}$, we have the following convergence for $\lambda \in C$ uniformly:

$$(\lambda - (\mathbf{M}_n + \mathbf{K}_n + \mathbf{S}_n))^{-1} f \underset{n\to\infty}{\longrightarrow} (\lambda - (\mathbf{M}_\infty + \mathbf{K}_\infty))^{-1} f \quad \textit{in } \mathcal{H} \tag{5.12}$$

The formula above can be proven by resolvent calculus, and the conclusion can be drawn by contour integration. □

6 Instability: From Euler to Navier-Stokes

As mentioned in the strategy part, the linearized operator of Navier-Stokes equation formally approaches the linearized operator of Euler equation when the amplitude of the velocity profile is sufficiently large. In this section, we will give a rigorous statement of the idea above.

6.1 Vorticity Instability

For l sufficiently large, we fix $\bar{U}$ to be the 3-dimensional axisymmetical-no-swirl velocity profile associated with $\bar{u}_l$ with respect to coordinate (r', z):

$$\bar{U}(r', z) := \bar{u}_l^r(r' - l, z)e_r + \bar{u}_l^z(r' - l, z)e_z$$

and $\bar{\Omega}$ is its corresponding vorticity field

Define the set of L^2 integrable axisymmetrical-pure-swirl vector field on $\mathbb{R}^3$:

$$L^2_{\text{aps}} = \{\Omega^\theta(r', z)e_\theta | \Omega^\theta \in L^2(\mathbb{R}^3)\}$$

We consider the base profile with relative amplitude β, i.e. the operator is linearized around $\beta\bar{U}$, take curl on the operator L_{ss} defined in (4.4) and restrict it on the set of L^2_{aps}. We obtain the following operator:

$$\begin{aligned} \mathbf{L}^{(\beta)}_{\text{vor}} : \mathrm{D}(\mathbf{L}^{(\beta)}_{\text{vor}}) &\to L^2_{\text{aps}} \\ \Omega &\mapsto \frac{1}{2}\left(2 + \xi\cdot\nabla_\xi\right)\Omega + \Delta\Omega - \beta([\bar{U}, \Omega] + [U, \bar{\Omega}]) \end{aligned}$$

where the domain of definition:

$$\mathrm{D}(\mathbf{L}^{(\beta)}_{\text{vor}}) := \{\Omega \in L^2_{\text{aps}} \cap H^2(\mathbb{R}^3) | \xi\cdot\nabla_\xi\Omega \in L^2(\mathbb{R}^3)\}$$

the lifted velocity field $U := \mathrm{BS}_{3d}[\Omega]$, and the bracket $[f, g] := (f\cdot\nabla)g - (g\cdot\nabla)f$.

We decompose the operator into the following terms and a multiplication by constant:

$$\begin{aligned} \mathbf{D}\Omega &= (\frac{3}{4} + \frac{\xi}{2}\cdot\nabla_\xi)\Omega + \Delta\Omega & \mathbf{M}\Omega &= -\bar{U}\cdot\nabla\Omega \\ \mathbf{S}\Omega &= \bar{\Omega}\cdot\nabla U + \Omega\cdot\nabla\bar{U} & \mathbf{K}\Omega &= -U\cdot\nabla\bar{\Omega} \end{aligned} \tag{6.1}$$

D is the **diffusion** term, **M** is the **main** term, **S** is the **small** term, and **K** is the **compact** term. Observe that D $-\Delta$ is a skew-adjoint operator.

Finally, we define:

$$\mathbf{T}_\beta := \frac{1}{\beta}\mathbf{D} + \mathbf{M} + \mathbf{S} + \mathbf{K} = \frac{1}{\beta}\left(\mathbf{L}_{\text{vor}}^{(\beta)} - \frac{1}{4}\right)$$

and we define the formal limit when $\beta \to \infty$:

$$\mathbf{T}_\infty := \mathbf{M} + \mathbf{S} + \mathbf{K}$$

where the domain of definition is

$$\mathrm{D}(\mathbf{T}_\infty) = \{\Omega \in L^2_{\text{aps}} | \bar{U} \cdot \nabla\Omega \in L^2\}$$

We view $\mathbf{T}_\infty$ as a lift of $\mathbf{L}_l$ into 3-dimensions, and the part **S** is exactly a part of $\mathbf{S}_l$ whose smallness can be obtained by slightly modifying the proof of Lemma 5.2. Thus, we can fix l sufficiently large, such that:

$$\alpha := \sup_{\lambda \in \sigma(\mathbf{T}_\infty)} \Re(\lambda) > \mu := \|\mathbf{S}\|_{L^2_{\text{aps}} \to \mathbf{L}^2(\mathbb{R}^3)} \tag{6.2}$$

From now on, we consider l and $\bar{U}$ as fixed and satisfying the estimate (6.2).

Now we state the instability of self-similar Navier-Stokes equation:

Theorem 6.1. *Let λ_∞ be an unstable eigenvalue of $\mathbf{T}_\infty$ such that $\Re(\lambda_\infty) > \mu$, then for all $\epsilon \in (0, \Re(\lambda_\infty) - \mu)$, there exists $\beta_0 > 0$, such that for all $\beta > \beta_0$, $\mathbf{T}_\beta$ has an unstable eigenvalue λ_β such that $|\lambda_\beta - \lambda_\infty| < \epsilon$, and $\mathbf{L}_{\text{vor}}^{(\beta)}$ has an unstable eigenvalue defined by*

$$\tilde{\lambda}_\beta := \beta\lambda_\beta + \frac{1}{4}$$

Sketch of Proof. we set ourselves in the setting of Lemma (5.3), let:

$$\mathbf{M}_\beta = \beta^{-\mathbf{1}}\mathbf{D} + \mathbf{M} + \mathbf{S}, \quad \mathbf{K}_\beta = \mathbf{K}, \quad \mathbf{S}_\beta = 0$$

It remains to verify the statements in the Lemma. □

6.2 Return to Velocity Form

In the following discussion, it's more convenient to work on the original linearized operator (4.4) which acts on the velocity fields, we introduce the amplified operator $L_{\text{ss}}^{(\beta)}$

$$\begin{aligned} \mathbf{L}_{\text{ss}}^{(\beta)} : \mathrm{D}(\mathbf{L}_{\text{ss}}) &\to L^2_\sigma \\ U &\mapsto \frac{1}{2}(1 + \xi \cdot \nabla_\xi) U + \Delta U - \beta\mathbb{P}(\bar{U} \cdot \nabla U + U \cdot \nabla\bar{U}) \end{aligned}$$

where we define the divergence-free subspace and the domain of definition:

$$L^2_\sigma = \{u \in L^2(\mathbb{R}^3)^3 | \nabla \cdot u = 0\},\ \mathrm{D}(\mathbf{L}_{\text{ss}}) = \{U \in L^2_\sigma \cap H^2(\mathbb{R}^3)^3 | \xi \cdot \nabla_\xi U \in L^2(\mathbb{R}^3)^3\}$$

The following corollary identifies the unstable eigenvalue of $\mathbf{L}_{\text{vor}}^{(\beta)}$ and $\mathbf{L}_{\text{ss}}$

Corollary 6.1. *In the setting of Theorem 6.1, for $\beta \geq \beta_0$, the unstable eigenvalue $\tilde{\lambda}_\beta$ is also an unstable eigenvalue of $\mathbf{L}_{\text{ss}}^{(\beta)}$.*

Sketch of Proof. By choosing β_0 sufficiently large, we can ensure that $\Re(\tilde{\lambda}_\beta) > 1$. Denoting the corresponding eigenfunction as $\Omega_\beta \in \mathrm{D}(\mathbf{L}_{\text{vor}}^{(\beta)})$, then we have:

$$(\tilde{\lambda}_\beta - 1)\Omega_\beta - \frac{\xi}{2} \cdot \nabla_\xi \Omega_\beta - \Delta\Omega_\beta = -\beta([\bar{U}, \Omega_\beta] + [U_\beta, \bar{\Omega}]) \tag{6.3}$$

We can see that $\mathrm{BS}_{3d}[\Omega_\beta]$ is the eigenvalue corresponding to $\tilde{\lambda}_\beta$ of $\mathbf{L}_{\text{ss}}^{(\beta)}$ as long as $\mathrm{BS}_{3d}[\Omega_\beta] \in \mathrm{D}(\mathbf{L}_{\text{ss}}^{(\beta)})$. □

7 Nonlinear Instability: Conclusion

In the previous paragraphs, we have studied the instability property of the linearized operator $\mathbf{L}_{\text{ss}}$ defined in (4.4) with a given background real, compactly supported velocity profile $\beta\bar{U}$ with β sufficiently large. Now, we want to apply this property into our self-similar equation (4.3), and obtain the asymptotic behavior when $\tau \to -\infty$.

In the following sections, we will use the notion introduced in (4.1) and (4.2) to separate the physical space and self-similar space and introduce the semigroup generated by the operator $\mathbf{L}_{\text{ss}}$ and finally construct U^{per} via fix point argument.

7.1 Study of the Linearized Operator

In this subsection, we want to consider the time evolution under the linearized operator. The spectrum of $\mathbf{L}_{\text{ss}}$ is bounded in the right and the norm of its resolvent is of order $O(\Re(z)^{-1})$. Applying Hille-Yosida's theorem, the operator generates a semigroup $e^{\tau\mathbf{L}_{\text{ss}}} : L^2_\sigma \to L^2_\sigma$. For a detailed proof, please refer to [JS13, Chapter 2].

We define the *right spectral bound* of $\mathbf{L}_{\text{ss}}$ as:

$$s(\mathbf{L}_{\text{ss}}) := \sup\{\Re(\lambda) | \lambda \in \sigma(\mathbf{L}_{\text{ss}})\} > 0$$

In the light of [EN01, Proposition 2.2], [Kat66, Corollary 2.11] and [JS13, Lemma 2.7], we conclude the spectral property by the proposition below:

Proposition 7.1. *We have $s(\mathbf{L}_{\text{ss}}) < \infty$, and there exists an eigenvalue $\lambda = s(\mathbf{L}_{\text{ss}}) + ib \in \sigma(\mathbf{L}_{\text{ss}})$ and its corresponding eigenfunction $\eta = \eta_1 + i\eta_2$, $\eta_i \in \mathrm{D}(\mathbf{L}_{\text{ss}})$ such that $\mathbf{L}_{\text{ss}}\eta = \lambda\eta$.*

Moreover, for $\delta > 0$, there exists $M(\delta) > 0$, such that:

$$\|e^{\tau\mathbf{L}_{\text{ss}}}\|_{L^2_\sigma \to L^2_\sigma} \leq M(\delta) e^{(s(\mathbf{L}_{\text{ss}})+\delta)\tau}$$

We take the linear perturbation to be

$$U^{\text{lin}}(\tau) = \Re(e^{\lambda\tau}\eta)$$

then it satisfies the linearized equation:

$$\partial_\tau U^{\text{lin}} = \mathbf{L}_{\text{ss}} U^{\text{lin}}$$

As there is a diffusion term in $\mathbf{L}_{\text{ss}}$, we expect parabolic regularity:

Proposition 7.2. *For any integer $\sigma_2 \geq \sigma_1 \geq 0$ and $\delta > 0$, then there exists $M = M(\sigma_1, \sigma_2, \delta)$, for all $U_0 \in L^2_\sigma \cap H^{\sigma_1}(\mathbb{R}^3)^3$, we have:*

$$\|e^{\tau \mathbf{L}_{\text{ss}}} U_0\|_{H^{\sigma_2}} \leq \frac{M}{\tau^{\frac{\sigma_2 - \sigma_1}{2}}} e^{(s(\mathbf{L}_{\text{ss}})+\delta)\tau} \|U_0\|_{H^{\sigma_1}}$$

Sketch of Proof. Observe that by a change of variables back into the physical space,

$$u(t,x) := \frac{1}{\sqrt{t+1}} U(\log(t+1), \frac{x}{\sqrt{t+1}}), \quad \bar{u}(t,x) := \frac{1}{\sqrt{t+1}} \bar{U}(\frac{x}{\sqrt{t+1}})$$

with $u(0,x) = U(0,x) =: U_0(x)$, then, we have the following equation:

$$\begin{cases} \partial_t u - \Delta u = -\mathbb{P}(\bar{u} \cdot \nabla u + u \cdot \nabla \bar{u}), & (t,x) \in [0,\infty) \times \mathbb{R}^3 \\ u(0,x) = U_0(x) \end{cases}$$

The proof essentially relies on the parabolic regularity for the heat semigroup:

$$t^{\frac{s}{2}} \|e^{t\Delta} f\|_{\dot{H}^s} \lesssim \|f\|_{L^2}, \quad \forall s \in \mathbb{N}$$

□

As a corollary, we have the eigenfunction $\eta \in H^k(\mathbb{R}^3)^3$ for all $k \in \mathbb{N}$, and,

$$\|U^{\text{lin}}(\tau)\|_{H^k} = \|\Re(e^{\tau \mathbf{L}_{\text{ss}}} \eta)\|_{H^k} = e^{s(\mathbf{L}_{\text{ss}})\tau} \|\Re(e^{ib\tau}\eta)\|_{H^k} \sim e^{s(\mathbf{L}_{\text{ss}})\tau} \tag{7.1}$$

where $\|\Re(e^{ib\tau}\eta)\|_{H^k}$ is a periodic function.

7.2 Construction of the Nonlinear Profile

In this subsection, we study the equation satisfied by the perturbation term in the self-similar variables:

$$\partial_\tau U^{\text{per}} + \mathbf{L}_{\text{ss}} U^{\text{per}} = -\mathbb{P}[((U^{\text{per}} + U^{\text{lin}}) \cdot \nabla)(U^{\text{per}} + U^{\text{lin}})] \tag{7.2}$$

with formal initial value $U^{\text{per}}(-\infty) = 0$.

We state the existence and decay of U^{per} in the proposition below:

Proposition 7.3. *Assume that $N > \frac{5}{2}$ is an integer, then, there exists lifespan $T = T(\bar{U}, U^{\text{lin}}, N) \in \mathbb{R}$, $\epsilon_0 = \epsilon_0(s(\mathbf{L}_{\text{ss}})) > 0$, such that (7.2) admits a solution $U^{\text{per}} \in C((-\infty, T], H^N(\mathbb{R}^3)^3)$, and we have the following estimate:*

$$\|U^{\text{per}}(\tau)\|_{H^N} \leq e^{(s(\mathbf{L}_{\text{ss}})+\epsilon_0)\tau} \quad \forall \tau \leq T$$

Let $T \in \mathbb{R}$ and $\epsilon_0 > 0$ to be two undetermined constants. we consider the Banach space equipped with norm decaying in time and the Duhamel functional:

$$X := \{U \in C((-\infty, T], H^N(\mathbb{R}^3)^3) | \|U\|_X := \sup_{\tau \leq T} e^{-(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)\tau} \|U\|_{H^N} < +\infty\}$$

$$\mathcal{T}U(\tau) := -\int_{-\infty}^{\tau} e^{(\tau-s)\mathbf{L}_{\mathrm{ss}}} \mathbb{P}[((U + U^{\mathrm{lin}}) \cdot \nabla)(U + U^{\mathrm{lin}})(s)] \mathrm{d}s$$

Proposition 7.4. *There exists $T = T(\bar{U}, U^{\mathrm{lin}}, N) \in \mathbb{R}$ and $\epsilon_0 = \epsilon_0(s(\mathbf{L}_{\mathrm{ss}})) > 0$, such that:*

$$\mathcal{T} : \{U \in X | \|U\|_X \leq 1\} \to \{U \in X | \|U\|_X \leq 1\}$$

is a contraction.

Sketch of Proof. The proposition essentially comes from the fact that for $M > \frac{3}{2}$, $H^M(\mathbb{R}^3)$ is a Banach algebra, i.e. for $f, g \in H^N(\mathbb{R}^3)$, we have:

$$\|fg\|_{H^M} \lesssim_M \|f\|_{H^M} \|g\|_{H^M}$$

Simply choose $\epsilon_0 = \frac{1}{2} s(\mathbf{L}_{\mathrm{ss}})$, T sufficient small will satisfy the proposition. □

Indeed, by a bootstrap argument, we have:

$$\sup_{\tau \in (-\infty, T]} e^{-2s(\mathbf{L}_{\mathrm{ss}})\tau} \|\mathcal{T}U(\tau)\|_{H^N} < \infty$$

Since U^{per} is a fixed point of $\mathcal{T}$, it satisfies the same estimate, thus the decay rate of $\|U^{\mathrm{per}}\|_{H^N}$ is actually $O(e^{2s(\mathbf{L}_{\mathrm{ss}})\tau})$.

Combining all the conclusions we have obtained, we have constructed the real, smooth, compactly supported unstable profile $\bar{U}$; the linear term U^{lin} associated to the most unstable eigenvalue, with decay of order $O(e^{s(\mathbf{L}_{\mathrm{ss}})\tau})$; and the perturbation term U^{per} with decay of order $O(e^{2s(\mathbf{L}_{\mathrm{ss}})\tau})$. The proof of Theorem 4.1 is done. Finally, rewrite the equation into the physical variables, with initial value $u|_{t=0} = 0$, Theorem 1.1 is proven.

References

[ABC22] Dallas Albritton, Elia Bruè, and Maria Colombo. Non-uniqueness of leray solutions of the forced navier-stokes equations. *Annals of Mathematics*, 196, 07 2022.

[BM20] Alberto Bressan and Ryan Murray. On self-similar solutions to the incompressible euler equations. *Journal of Differential Equations*, 269(6):5142–5203, 2020.

[BS21] Alberto Bressan and Wen Shen. A posteriori error estimates for self-similar solutions to the euler equations. *Discrete and Continuous Dynamical Systems*, 41(1):113–130, 2021.

[BV19] Tristan Buckmaster and Vlad Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Annals of Mathematics*, 189(1):101 – 144, 2019.

[EN01] Klaus-Jochen Engel and Rainer Nagel. One-parameter semigroups for linear evolution equations. *Semigroup Forum*, 63:278–280, 06 2001.

[JS13] Hao Jia and Vladimir Sverak. Are the incompressible 3d navier-stokes equations locally ill-posed in the natural energy space? *arXiv: Analysis of PDEs*, 2013.

[Kat66] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer, 1966.

[Ler34] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica*, 63(none):193 – 248, 1934.

[LP59] Jacques-Louis Lions and Giovanni Prodi. Un théorème d'existence et unicité dans les équations de Navier- Stokes en dimension 2. *C. R. Acad. Sci., Paris*, 248:3519–3521, 1959.

[Vis18a] Misha Vishik. Instability and non-uniqueness in the Cauchy problem for the Euler equations of an ideal incompressible fluid. Part I. *arXiv e-prints*, page arXiv:1805.09426, May 2018.

[Vis18b] Misha Vishik. Instability and non-uniqueness in the Cauchy problem for the Euler equations of an ideal incompressible fluid. Part II. *arXiv e-prints*, page arXiv:1805.09440, May 2018.